

Orlicz-Legendre Ellipsoids

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Abstract The Orlicz-Legendre ellipsoids, which are in the framework of emerging dual Orlicz Brunn-Minkowski theory, are introduced for the first time. They are in some sense dual to the recently found Orlicz-John ellipsoids, and have largely generalized the classical Legendre ellipsoid of inertia. Several new affine isoperimetric inequalities are established. The connection between the characterization of Orlicz-Legendre ellipsoids and isotropy of measures is demonstrated.

2000 Mathematics Subject Classification: 52A40.

Keywords: Orlicz Brunn-Minkowski theory; Legendre ellipsoid; Löwner ellipsoid; Isotropy

1. Introduction

Corresponding to each body in Euclidean n -space \mathbb{R}^n , there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the body are the same about *every* 1-dimensional subspace of \mathbb{R}^n . This ellipsoid is called the *Legendre ellipsoid* of the body. The Legendre ellipsoid is a well-known concept from classical mechanics, and is closely related with the long-standing unsolved maximal slicing problem. See, e.g., Lindenstrauss and Milman [34], and Milman and Pajor [53].

The Legendre ellipsoid is an object in the *dual* Brunn-Minkowski theory, which was originated by Lutwak [38] and achieved great developments since 1980s. See, e.g., [11, 14, 15, 17, 39, 40, 65, 66]. It is remarkable that for each convex body (compact convex subset with non-empty interior) K in \mathbb{R}^n , Lutwak, Yang and Zhang [43] introduced a new ellipsoid by using the notion of L_2 -curvature, which is now called the *LYZ ellipsoid* and is precisely the dual analogue of the Legendre ellipsoid.

Following LYZ [43], we write $\Gamma_2 K$ and $\Gamma_{-2} K$ for the Legendre ellipsoid and LYZ ellipsoid, respectively. In [46], LYZ extended the domain of Γ_{-2} to star-shaped sets and showed

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Research of the authors was supported by NSFC No. 11471206.

the relationship between the two ellipsoids: If K is a star-shaped set, then $\Gamma_{-2}K \subset \Gamma_2K$, with equality if and only if K is an ellipsoid centered at the origin. This inclusion is the geometric analogue of one of the basic inequalities in information theory - the *Cramer-Rao inequality*. When viewed as suitably normalized matrix-valued operators on the space of convex bodies, it was proved by Ludwig [35] that the Legendre ellipsoid and the LYZ ellipsoid are the only linearly invariant operators that satisfy the inclusion-exclusion principle. The Legendre ellipsoid has also applications in Finsler geometry [52].

In the geometry of convex bodies, many extremal problems of an affine nature often have ellipsoids as extremal bodies. Besides the above mentioned Legendre ellipsoid and LYZ ellipsoid, the *John ellipsoid* JK [30] and the *Löwner ellipsoid* LK are of fundamental importance. Since the object considered in this paper is *dual* to the John ellipsoid, in what follows, we recall the John ellipsoid in detail.

Associated with each convex body K in \mathbb{R}^n , its John ellipsoid JK is the unique ellipsoid of maximal volume contained in K . The John ellipsoid has many applications in convex geometry, functional analysis, PDEs, etc. Particularly, by combining the isotropic characterization of the John ellipsoid and the celebrated Brascamp-Lieb inequality, it has powerful effect on attacking reverse isoperimetric problems. See, e.g., [1, 2, 47, 48, 49, 60].

Since 2005, the family of John ellipsoid has expanded rapidly, and experienced the L_p stage [48] and the very recent Orlicz stage [68]. It is interesting that with the expansion of the family, several ellipsoids, including the LYZ ellipsoid, are found to be close relatives of the John ellipsoid. We do a bit review on this point.

Motivated by the study of geometry of L_p Brunn-Minkowski theory (See, e.g., [41, 42, 44]), LYZ [48] introduced a family of ellipsoids, called the L_p John ellipsoids E_pK , $p > 0$. It is striking that the bodies E_pK form a *spectrum* linking several fundamental objects in convex geometry: If the John point of K , i.e., the center of JK , is at the origin, then $E_\infty K$ is precisely the classical John ellipsoid JK . The L_2 John ellipsoid E_2K is just the LYZ ellipsoid. The L_1 John ellipsoid E_1K is the so-called *Petty ellipsoid*. The volume-normalized Petty ellipsoid is obtained by minimizing the surface area of K under $SL(n)$ transformations of K ([20, 55]).

Throughout this paper, we consider convex $\varphi : [0, \infty) \rightarrow [0, \infty)$, that is strictly increasing and satisfies $\varphi(0) = 0$. Along the line of extension, the authors of this paper originally introduced the *Orlicz-John ellipsoids* [68] $E_\varphi K$ for each convex body K with the origin in its interior, in the framework of booming Orlicz Brunn-Minkowski theory (See, e.g., [18, 19, 27, 36, 50, 51]). The new Orlicz-John ellipsoids $E_\varphi K$ generalize LYZ's L_p John ellipsoids E_pK to the Orlicz setting, analogous to the way that Orlicz norms [58] generalize L_p norms. Indeed, If $\varphi(t) = t^p$, $1 \leq p < \infty$, then $E_\varphi K$ precisely turns to the L_p John ellipsoid E_pK . If $p \rightarrow \infty$, then $E_{\varphi^p}K$ approaches to $E_\infty K$.

The *Löwner ellipsoid* LK is the unique ellipsoid of minimal volume containing K , which is investigated widely in the field of convex geometry and local theory of Banach spaces. We refer to, e.g., [1, 2, 12, 20, 23, 24, 25, 26, 30, 32, 33, 34, 37, 57, 67].

As LYZ [46] pointed out, there is in fact a “dictionary” correspondence between the Brunn-Minkowski theory and its dual. In retrospect, the John ellipsoid, LYZ ellipsoid and Petty ellipsoid are objects within the Brunn-Minkowski theory; while the Legendre ellipsoid and Löwner ellipsoid are objects within the dual Brunn-Minkowski theory. Along the idea of dictionary relation, we are tempted to consider the naturally posed problem: What is the dual analogue of the newly found Orlicz-John ellipsoid?

One of the main task in this paper is to demonstrate the existence of such a *dual* analogue of Orlicz-John ellipsoid. Incidentally, it precisely acts as the *spectrum* linking the Legendre ellipsoid and Löwner ellipsoid. So, this paper is a sequel of [68].

For star bodies K, L in \mathbb{R}^n , define the *normalized dual Orlicz mixed volume* $\tilde{V}_\varphi(K, L)$ of K and L with respect to φ by

$$\tilde{V}_\varphi(K, L) = \varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\rho_L} \right) dV_K^* \right).$$

Here, S^{n-1} is the unit sphere in \mathbb{R}^n ; ρ_K and ρ_L are the radial functions of K and L , respectively; V_K^* is the normalized dual conical measure of K , defined by

$$dV_K^* = \frac{\rho_K^n}{nV(K)} dS,$$

where S is the spherical Lebesgue measure on S^{n-1} .

Enlightened by our work on Orlicz-John ellipsoids [68], we focus on

Problem \tilde{S}_φ . *Suppose K is a star body in \mathbb{R}^n . Find an ellipsoid E , amongst all origin-symmetric ellipsoids, which solves the following constrained minimization problem:*

$$\min_E V(E) \quad \text{subject to} \quad \tilde{V}_\varphi(K, E) \leq 1.$$

In Section 4, we prove that there exists a unique ellipsoid which solves the above minimization problem. It is called the *Orlicz-Legendre ellipsoid* of K with respect to φ , and denoted by $L_\varphi K$. If $\varphi(t) = t^2$, then $L_\varphi K$ is precisely the Legendre ellipsoid $\Gamma_2 K$.

It is interesting that the Orlicz-Legendre ellipsoid mirrors the Orlicz-John ellipsoid.

Similar to the important property of Orlicz-John ellipsoid $E_\varphi K$, in Section 5 we show that the Orlicz-Legendre ellipsoid $L_\varphi K$ is jointly continuous in φ and K . In Section 6, it is proved that as $p \rightarrow \infty$, $L_{\varphi^p} K$ approaches to a common ellipsoid $L_\infty K$, the unique ellipsoid of minimal volume containing K . This insight throws light on a connection between Orlicz-Legendre ellipsoids and the Löwner ellipsoid.

In Section 7, we establish a characterization of Orlicz-Legendre ellipsoids, which is closely related to the isotropy of measures.

In general, Orlicz-Legendre ellipsoids $L_\varphi K$ do not contain K . In Section 8, we prove that: If K is a star body (about the origin) in \mathbb{R}^n , then

$$V(L_\varphi K) \geq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

If $\varphi(t) = t^2$, it reduces to the celebrated inequality: $V(\Gamma_2 K) \geq V(K)$, which goes back to Blaschke [6], John [29], Milman and Pajor [53], Petty [56], and also LYZ [43].

2. Preliminaries

2.1. Notations. The setting will be the Euclidean n -space \mathbb{R}^n . As usual, $x \cdot y$ denotes the standard inner product of x and y in \mathbb{R}^n , and V denotes the n -dimensional volume.

In addition to its denoting absolute value, without confusion we often use $|\cdot|$ to denote the standard Euclidean norm, on occasion the total mass of a measure, and the absolute value of the determinant of an $n \times n$ matrix.

For a continuous real function f defined on S^{n-1} , write $\|f\|_\infty$ for the L_∞ norm of f . Let \mathcal{L}^n denote the space of linear operators from \mathbb{R}^n to \mathbb{R}^n . For $T \in \mathcal{L}^n$, T^t and $\|T\|$ denote the transpose and norm of T , respectively.

A finite positive Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\frac{n}{|\mu|} \int_{S^{n-1}} (u \cdot v)^2 d\mu(u) = 1, \quad \text{for all } v \in S^{n-1}.$$

For nonzero $x \in \mathbb{R}^n$, the notation $x \otimes x$ represents the rank 1 linear operator on \mathbb{R}^n that takes y to $(x \cdot y)x$. It immediately gives

$$\text{tr}(x \otimes x) = |x|^2.$$

Equivalently, μ is isotropic if

$$\frac{n}{|\mu|} \int_{S^{n-1}} u \otimes u d\mu(u) = I_n,$$

where I_n denotes the identity operator on \mathbb{R}^n . For more information on the isotropy of measures, we refer to [5, 20, 21, 53].

2.2. Orlicz norms. Throughout this paper, Φ denotes the class of convex functions $\varphi : [0, \infty) \rightarrow [0, \infty)$, that are strictly increasing and satisfy $\varphi(0) = 0$.

We say a sequence $\{\varphi_i\}_{i \in \mathbb{N}} \subset \Phi$ is such that $\varphi_i \rightarrow \varphi_0 \in \Phi$, provided

$$|\varphi_i - \varphi_0|_I := \max_{t \in I} |\varphi_i(t) - \varphi_0(t)| \rightarrow 0,$$

for each compact interval $I \subset [0, \infty)$.

Let μ be a finite positive Borel measure on S^{n-1} . For a continuous function $f : S^{n-1} \rightarrow [0, \infty)$, the *Orlicz norm* $\|f : \mu\|_\varphi$ of f , is defined by

$$\|f : \mu\|_\varphi = \inf \left\{ \lambda > 0 : \frac{1}{|\mu|} \int_{S^{n-1}} \varphi \left(\frac{f}{\lambda} \right) d\mu \leq \varphi(1) \right\}.$$

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then $\|f : \mu\|_\varphi$ is just the classical L_p norm. According to the context, without confusion we write $\|f\|_\varphi$ for $\|f : \mu\|_\varphi$.

Lemma 2.1 was previously proved in [27], which will be used frequently.

Lemma 2.1. *Suppose μ is a finite positive Borel measure on S^{n-1} and the function $f : S^{n-1} \rightarrow [0, \infty)$ is continuous and such that $\mu(\{f \neq 0\}) > 0$. Then the function*

$$\psi(\lambda) := \int_{S^{n-1}} \varphi \left(\frac{f}{\lambda} \right) d\mu, \quad \lambda \in (0, \infty),$$

has the following properties:

- (1) ψ is continuous and strictly decreasing in $(0, \infty)$;
- (2) $\lim_{\lambda \rightarrow 0^+} \psi(\lambda) = \infty$;
- (3) $\lim_{\lambda \rightarrow \infty} \psi(\lambda) = 0$;
- (4) $0 < \psi^{-1}(a) < \infty$ for each $a \in (0, \infty)$.

Consequently, the Orlicz norm $\|f\|_\varphi$ is strictly positive. Moreover,

$$\|f\|_\varphi = \lambda_0 \iff \frac{1}{|\mu|} \int_{S^{n-1}} \varphi \left(\frac{f}{\lambda_0} \right) d\mu = \varphi(1).$$

2.3. Convex bodies and star bodies. The *support function* h_K of a compact convex set K in \mathbb{R}^n is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad \text{for } x \in \mathbb{R}^n.$$

For $T \in \text{GL}(n)$, the support function of the image $TK = \{Tx : x \in K\}$ is given by

$$h_{TK}(x) = h_K(T^t x).$$

As usual, a *body* is a compact set with non-empty interior. Write \mathcal{K}_o^n for the class of convex bodies in \mathbb{R}^n that contain the origin in their interiors. \mathcal{K}_o^n is often equipped with the *Hausdorff metric* δ_H , which is defined by

$$\delta_H(K_1, K_2) = \max\{|h_{K_1}(u) - h_{K_2}(u)| : u \in S^{n-1}\}, \quad \text{for } K_1, K_2 \in \mathcal{K}_o^n.$$

That is

$$\delta_H(K_1, K_2) = \|h_{K_1} - h_{K_2}\|_\infty.$$

Next, we turn to some basics on star bodies.

A set $K \subseteq \mathbb{R}^n$ is *star-shaped*, if $\lambda x \in K$ for $\forall(\lambda, x) \in [0, 1] \times K$. For a non-empty, compact and star-shaped set K in \mathbb{R}^n , its *radial function* ρ_K is defined by

$$\rho_K(x) = \sup\{\lambda \geq 0 : \lambda x \in K\}, \quad \text{for } x \in \mathbb{R}^n \setminus \{o\}.$$

It is easily seen that ρ_K is homogeneous of degree -1 . For $T \in \text{GL}(n)$, we obviously have

$$(2.1) \quad \rho_{TK}(x) = \rho_K(T^{-1}x).$$

A star-shaped set K is called a *star body* about the origin o , if $o \in \text{int}K$, and its radial function ρ_K is continuous on S^{n-1} . Write \mathcal{S}_o^n for the class of star bodies about the origin o in \mathbb{R}^n . \mathcal{S}_o^n is often equipped with the *dual Hausdorff metric* $\tilde{\delta}_H$, which is defined by

$$\tilde{\delta}_H(K_1, K_2) = \max\{|\rho_{K_1}(u) - \rho_{K_2}(u)| : u \in S^{n-1}\}, \quad \text{for } K_1, K_2 \in \mathcal{S}_o^n.$$

That is,

$$\tilde{\delta}_H(K_1, K_2) = \|\rho_{K_1} - \rho_{K_2}\|_\infty.$$

The *dual conical measure* \tilde{V}_K , of a star body $K \in \mathcal{S}_o^n$, is a Borel measure on S^{n-1} defined by

$$d\tilde{V}_K = \frac{\rho_K^n}{n} dS.$$

It is convenient to use its normalization V_K^* , given by $V_K^* = \frac{\tilde{V}_K}{V(K)}$. Observe that V_K^* was firstly introduced by LYZ [51] to define Orlicz centroid bodies. Note that the dual conical measure differs from the cone-volume measure (See, e.g., [7, 8, 27, 28, 50, 61, 62]), but both are outgrowth from the cone measure (See, e.g. [4, 22, 54]).

Note that for each Borel subset $\omega \subseteq S^{n-1}$, we also have

$$\tilde{V}_K(\omega) = V(K \cap \{su : s \geq 0 \text{ and } u \in \omega\}).$$

Thus, it follows that

$$(2.2) \quad \tilde{V}_{TK}(\omega) = \tilde{V}_K(\langle T^{-1}\omega \rangle), \quad \text{for } T \in \text{SL}(n),$$

where $\langle T^{-1}\omega \rangle = \{\frac{T^{-1}u}{|T^{-1}u|} : u \in \omega\}$.

For $K \in \mathcal{K}_o^n$, its *polar body* K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for } y \in K\}.$$

For $K \in \mathcal{K}_o^n$, we have

$$(2.3) \quad \rho_{K^*}(u) = \frac{1}{h_K(u)} \quad \text{and} \quad h_{K^*}(u) = \frac{1}{\rho_K(u)}, \quad \text{for } u \in S^{n-1},$$

and

$$(2.4) \quad (TK)^* = T^{-t}K^*, \quad \text{for } T \in \text{GL}(n).$$

2.4. Ellipsoids and linear operators. Throughout, \mathcal{E}^n is used exclusively to denote the class of n -dimensional origin-symmetric ellipsoids in \mathbb{R}^n .

For $E \in \mathcal{E}^n$, let $d(E)$ denote its maximal principal radius. Two facts are in order. First, $T \in \mathcal{L}^n$ is non-degenerated, if and only if the ellipsoid TB is non-degenerated. Second, for $T \in \mathcal{L}^n$, since

$$\|T\| = \max_{u \in S^{n-1}} |Tu| = \max_{u \in S^{n-1}} |T^t u| = \|T^t\|,$$

it follows that

$$d(TB) = \max_{u \in S^{n-1}} h_{TB}(u) = \max_{u \in S^{n-1}} |T^t u| = \max_{u \in S^{n-1}} |Tu| = \max_{u \in S^{n-1}} h_{T^t B}(u) = d(T^t B).$$

Let

$$d_n(T_1, T_2) = \|T_1 - T_2\|, \quad \text{for } T_1, T_2 \in \mathcal{L}^n.$$

Then the metric space (\mathcal{L}^n, d_n) is complete. Since \mathcal{L}^n is of finite dimension, a set in (\mathcal{L}^n, d_n) is compact, if and only if it is bounded and closed.

We conclude this section with three lemmas, which will be used in Sections 4 - 6. For their proofs, we refer to Appendix A.

Lemma 2.2. *Suppose $\{T_j\}_{j \in \mathbb{N}} \subset \text{SL}(n)$. Then*

$$\|T_j\| \rightarrow \infty \iff \|T_j^{-1}\| \rightarrow \infty.$$

Thus, $\{T_j\}_{j \in \mathbb{N}}$ is bounded from above, if and only if $\{T_j^{-1}\}_{j \in \mathbb{N}}$ is bounded from above.

Lemma 2.3. *Suppose $\{T_j\}_{j \in \mathbb{N}} \subset \text{SL}(n)$, and $T_j \rightarrow T_0 \in \text{SL}(n)$ with respect to d_n . Then*

- (1) $T_j^t B \rightarrow T_0^t B$ with respect to δ_H .
- (2) $T_j^{-1} \rightarrow T_0^{-1}$ with respect to d_n .
- (3) $T_j B \rightarrow T_0 B$ with respect to $\tilde{\delta}_H$.

Lemma 2.4. *Suppose $E_0 \in \mathcal{E}^n$, $\{E_j\}_{j \in \mathbb{N}} \subset \mathcal{E}^n$ and $V(E_j) = a$, $\forall j \in \mathbb{N}$, $a > 0$. Then $E_j \rightarrow E_0$ with respect to δ_H , if and only if $E_j \rightarrow E_0$ with respect to $\tilde{\delta}_H$.*

3. Dual Orlicz mixed volumes

In order to define Orlicz-Legendre ellipsoids, we make some necessary preparations.

Definition 3.1. Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. The geometric quantity

$$\tilde{V}_\varphi(K, L) := \int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\rho_L}\right) d\tilde{V}_K$$

is called the *dual Orlicz mixed volume* of K and L with respect to φ . The quantity

$$\tilde{V}_\varphi(K, L) := \varphi^{-1} \left(\frac{\tilde{V}_\varphi(K, L)}{V(K)} \right) = \varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\rho_L} \right) dV_K^* \right)$$

is called the *normalized dual Orlicz mixed volume* of K and L with respect to φ .

Obviously, $\tilde{V}_\varphi(K, K) = \varphi(1)V(K)$, and $\tilde{\tilde{V}}(K, K) = 1$.

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then $\tilde{V}_\varphi(K, L)$ reduces to the classical dual mixed volume

$$\tilde{V}_{-p}(K, L) = \int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L} \right)^p d\tilde{V}_K,$$

and $\tilde{\tilde{V}}_\varphi(K, L)$ reduces to normalized dual mixed volume [64]

$$\tilde{\tilde{V}}_{-p}(K, L) := \left[\frac{\tilde{V}_{-p}(K, L)}{V(K)} \right]^{\frac{1}{p}} = \left(\int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L} \right)^p dV_K^* \right)^{\frac{1}{p}}.$$

Lemma 3.2. *Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

- (1) $\tilde{V}_\varphi(TK, L) = |T| \tilde{V}_\varphi(K, T^{-1}L)$, for $T \in \text{GL}(n)$.
- (2) $\tilde{\tilde{V}}_\varphi(TK, L) = \tilde{\tilde{V}}_\varphi(K, T^{-1}L)$, for $T \in \text{GL}(n)$.
- (3) $\tilde{\tilde{V}}_\varphi(\lambda K, L) = \tilde{\tilde{V}}_\varphi(K, \lambda^{-1}L)$, for $\lambda > 0$.

Proof. Suppose $T \in \text{GL}(n)$. For $u \in S^{n-1}$, let $\langle T^{-1} \rangle = T^{-1}u/|T^{-1}u|$. From Definition 3.1, (2.1) and (2.2), it follows that

$$\begin{aligned} \tilde{V}_\varphi(TK, L) &= \int_{S^{n-1}} \varphi \left(\frac{\rho_{TK}(u)}{\rho_L(u)} \right) d\tilde{V}_{TK}(u) \\ &= |T| \int_{S^{n-1}} \varphi \left(\frac{\rho_K(\langle T^{-1}u \rangle)}{\rho_{T^{-1}L}(\langle T^{-1}u \rangle)} \right) d\tilde{V}_K(\langle T^{-1}u \rangle) \\ &= |T| \int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\rho_{T^{-1}L}} \right) d\tilde{V}_K \\ &= |T| \tilde{V}_\varphi(K, T^{-1}L), \end{aligned}$$

as desired.

From (1) and Definition 3.1, we have

$$\tilde{\tilde{V}}_\varphi(TK, L) = \varphi^{-1} \left(\frac{\tilde{V}_\varphi(TK, L)}{V(TK)} \right) = \varphi^{-1} \left(\frac{\tilde{V}_\varphi(K, T^{-1}L)}{V(K)} \right) = \tilde{\tilde{V}}_\varphi(K, T^{-1}L),$$

as desired.

Take $T = \lambda I_n$ in (2), it yields (3) directly. □

Along with the functional $\tilde{V}_\varphi(K, L)$, we introduce

Definition 3.3. Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$, define

$$O_\varphi(K, L) = \left\| \frac{\rho_K}{\rho_L} : \tilde{V}_K \right\|_\varphi = \inf \left\{ \lambda > 0 : \varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right) dV_K^* \right) \leq 1 \right\}.$$

Obviously, $O_\varphi(K, K) = 1$. If $\varphi(t) = t^p$, $1 \leq p < \infty$, then $O_\varphi(K, L) = \tilde{V}_{-p}(K, L)$.

From Definition 3.3 and Definition 3.1, we have

$$\begin{aligned} O_\varphi(K, L) &= \inf \left\{ \lambda > 0 : \frac{\tilde{V}_\varphi(K, \lambda L)}{V(K)} \leq \varphi(1) \right\} \\ &= \inf \left\{ \lambda > 0 : \tilde{V}_\varphi(K, \lambda L) \leq 1 \right\}. \end{aligned}$$

Combining this with Lemma 3.2, we immediately obtain

Lemma 3.4. Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then

- (1) $O_\varphi(TK, L) = O_\varphi(K, T^{-1}L)$, for all $T \in \text{GL}(n)$.
- (2) $O_\varphi(\lambda K, L) = O_\varphi(K, \lambda^{-1}L) = \lambda O_\varphi(K, L)$, for all $\lambda > 0$.

The next lemma provides a simple but powerful identity.

Lemma 3.5. Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then

$$\tilde{V}_\varphi(K, O_\varphi(K, L)L) = 1.$$

Consequently, there is the following equivalence

$$\tilde{V}_\varphi(K, L) = 1 \iff O_\varphi(K, L) = 1.$$

Proof. From Definition 3.1, Definition 3.3, together with Lemma 2.1, it follows that

$$\varphi \left(\tilde{V}_\varphi(K, O_\varphi(K, L)L) \right) = \int_{S^{n-1}} \varphi \left(\frac{\rho_K}{O_\varphi(K, L)\rho_K} \right) dV_K^* = \varphi(1).$$

Thus, $\tilde{V}_\varphi(K, O_\varphi(K, L)L) = 1$. By Lemma 2.1 again, the desired equivalence follows. \square

What follows establishes the dual Orlicz Minkowski inequalities.

Lemma 3.6. Suppose $K, L \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then

$$(3.1) \quad \tilde{V}_\varphi(K, L) \geq \left(\frac{V(K)}{V(L)} \right)^{\frac{1}{n}},$$

and

$$(3.2) \quad O_\varphi(K, L) \geq \left(\frac{V(K)}{V(L)} \right)^{\frac{1}{n}}.$$

Each equality holds in the above inequalities if and only if K and L are dilates.

Proof. From Definition 3.1, the fact that φ^{-1} is strictly increasing in $(0, \infty)$ together with the convexity of φ and Jensen's inequality, the definition of V_K^* , and the reverse Hölder inequality, we have

$$\begin{aligned}
\tilde{V}_\varphi(K, L) &= \varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\rho_L} \right) dV_K^* \right) \\
&\geq \varphi^{-1} \left(\varphi \left(\int_{S^{n-1}} \frac{\rho_K}{\rho_L} dV_K^* \right) \right) \\
&= \frac{1}{nV(K)} \int_{S^{n-1}} \frac{\rho_K^{n+1}}{\rho_L} dS \\
&\geq \frac{1}{V(K)} \left(\frac{1}{n} \int_{S^{n-1}} \rho_K^{(n+1) \cdot \frac{n}{n+1}} dS \right)^{\frac{n+1}{n}} \left(\frac{1}{n} \int_{S^{n-1}} \rho_L^{-1 \cdot (-n)} dS \right)^{-\frac{1}{n}} \\
&= \left(\frac{V(K)}{V(L)} \right)^{\frac{1}{n}}.
\end{aligned}$$

By the equality condition of the reverse Hölder inequality, we know that the equality in the forth line occurs only if ρ_K/ρ_L is a positive constant on S^{n-1} . Thus, the equality holds in (3.1) only if K and L are dilates. Conversely, if $K = sL$ for some $s > 0$, then $\tilde{V}_\varphi(K, L) = s = (V(K)/V(L))^{1/n}$.

Combining Lemma 3.5 with inequality (3.1), we have

$$1 = \tilde{V}_\varphi(K, O_\varphi(K, L)L) \geq \left(\frac{V(K)}{V(O_\varphi(K, L)L)} \right)^{\frac{1}{n}} = \frac{1}{O_\varphi(K, L)} \left(\frac{V(K)}{V(L)} \right)^{\frac{1}{n}},$$

where the equality holds if and only if K and $O_\varphi(K, L)L$ are dilates. Thus, inequality (3.2), as well as its equality condition, is derived. \square

The next lemma is crucial to prove the continuity of the functionals $\tilde{V}_\varphi(K, L)$, $\tilde{V}_\varphi(K, L)$ and $O_\varphi(K, L)$ in (K, L, φ) .

Lemma 3.7. *Suppose f_i, f are strictly positive and continuous functions on S^{n-1} ; $\varphi_k, \varphi \in \Phi$; μ_l, μ are Borel probability measures on S^{n-1} ; $i, k, l \in \mathbb{N}$. If $f_i \rightarrow f$ pointwise, $\varphi_k \rightarrow \varphi$, and $\mu_l \rightarrow \mu$ weakly, then*

$$(3.3) \quad \int_{S^{n-1}} \varphi_k(f_i) d\mu_l \rightarrow \int_{S^{n-1}} \varphi(f) d\mu,$$

$$(3.4) \quad \varphi_k^{-1} \left(\int_{S^{n-1}} \varphi_k(f_i) d\mu_l \right) \rightarrow \varphi^{-1} \left(\int_{S^{n-1}} \varphi(f) d\mu \right),$$

and

$$(3.5) \quad \|f_i : \mu_l\|_{\varphi_k} \rightarrow \|f : \mu\|_\varphi.$$

Proof. The continuity of f_i and f , and $f_i \rightarrow f$ pointwise guarantee that $f_i \rightarrow f$ uniformly. Thus, there exists an $N_0 \in \mathbb{N}$, such that

$$\frac{1}{2} \min_{u \in S^{n-1}} f(u) \leq f_i \leq 2 \max_{u \in S^{n-1}} f(u), \quad \text{for } i > N_0.$$

Let

$$c_m = \min \left\{ \frac{1}{2} \min_{u \in S^{n-1}} f(u), \min_{u \in S^{n-1}} f_i(u), \text{ with } i \leq N_0 \right\},$$

and

$$c_M = \max \left\{ 2 \max_{u \in S^{n-1}} f(u), \max_{u \in S^{n-1}} f_i(u), \text{ with } i \leq N_0 \right\}.$$

The strictly positivity and the continuity of f_i and f imply that

$$0 < c_m \leq c_M < \infty.$$

Thus,

$$(3.6) \quad c_m \leq f(u) \leq c_M \quad \text{and} \quad c_m \leq f_i(u) \leq c_M, \quad \text{for } u \in S^{n-1} \text{ and } i \in \mathbb{N}.$$

Since $\varphi_k \rightarrow \varphi$ uniformly on $[c_m, c_M]$, by (3.6) and that $f_i \rightarrow f$ uniformly, it follows that as $i, k \rightarrow \infty$,

$$\varphi_k(f_i) \rightarrow \varphi(f), \quad \text{uniformly on } S^{n-1}.$$

Combined with that $\mu_l \rightarrow \mu$ weakly, it concludes that as $i, k, l \rightarrow \infty$,

$$\int_{S^{n-1}} \varphi_k(f_i) d\mu_l \rightarrow \int_{S^{n-1}} \varphi(f) d\mu,$$

as (3.3) desired.

Now, we proceed to prove (3.4).

For brevity, let

$$a_{i,k,l} = \int_{S^{n-1}} \varphi_k(f_i) d\mu_l \quad \text{and} \quad a = \int_{S^{n-1}} \varphi(f) d\mu.$$

Then

$$\varphi(c_m) \leq a \leq \varphi(c_M) \quad \text{and} \quad \varphi_k(c_m) \leq a_{i,k,l} \leq \varphi_k(c_M), \quad \text{for } i, k, l \in \mathbb{N}.$$

Let

$$a_m = \inf \{ \varphi(c_m), \varphi_k(c_m), \text{ with } k \in \mathbb{N} \},$$

and

$$a_M = \sup \{ \varphi(c_M), \varphi_k(c_M), \text{ with } k \in \mathbb{N} \}.$$

Since $\varphi_k(c_m) \rightarrow \varphi(c_m)$ and $\varphi_k(c_M) \rightarrow \varphi(c_M)$, it gives

$$0 < a_m \leq a_M < \infty \quad \text{and} \quad a, a_{i,k,l} \in [a_m, a_M], \quad \text{for } i, k, l \in \mathbb{N}.$$

Since $\varphi_k \rightarrow \varphi$ uniformly on $[c_m, c_M]$, it follows that

$$\varphi_k^{-1} \rightarrow \varphi^{-1}, \quad \text{uniformly on } [a_m, a_M].$$

Thus, from that $a_{i,k,l} \rightarrow a$ as $i, k, l \rightarrow \infty$, it follows that

$$\varphi_k^{-1}(a_{i,k,l}) \rightarrow \varphi^{-1}(a), \quad \text{as } i, k, l \rightarrow \infty,$$

as (3.4) desired.

Finally, we conclude to show (3.5).

At first, we prove that the set $\{\|f_i : \mu_l\|_{\varphi_k} : i, k, l \in \mathbb{N}\}$ is bounded.

Indeed, from (3.6) together with the strict monotonicity of φ and φ^{-1} , Lemma 2.1, and (3.6) together with the strict monotonicity of φ and φ^{-1} again, it follows that

$$\begin{aligned} \frac{c_m}{\|f_i : \mu_l\|_{\varphi_k}} &\leq \varphi_k^{-1} \left(\int_{S^{n-1}} \varphi_k \left(\frac{f_i}{\|f_i : \mu_l\|_{\varphi_k}} \right) d\mu_l \right) \\ &= 1 \\ &\leq \frac{c_M}{\|f_i : \mu_l\|_{\varphi_k}}, \end{aligned}$$

which immediately gives

$$c_m \leq \|f_i : \mu_l\|_{\varphi_k} \leq c_M, \quad \text{for } i, k, l \in \mathbb{N}.$$

Now, we can complete the proof of (3.5).

Since $\{\|f_i : \mu_l\|_{\varphi_k} : i, k, l \in \mathbb{N}\}$ is bounded, to prove (3.5), it suffices to prove that each convergent subsequence $\{\|f_{i_p} : \mu_{l_r}\|_{\varphi_{k_q}}\}_{p,q,r \in \mathbb{N}}$ of $\{\|f_i : \mu_l\|_{\varphi_k} : i, k, l \in \mathbb{N}\}$ necessarily converges to $\|f : \mu\|_{\varphi}$, as $i_p, k_q, l_r \rightarrow \infty$.

Assume

$$\lim_{p,q,r \rightarrow \infty} \|f_{i_p} : \mu_{l_r}\|_{\varphi_{k_q}} = \lambda_0.$$

Note that

$$\frac{f_{i_p}}{\|f_{i_p} : \mu_{l_r}\|_{\varphi_{k_q}}} \rightarrow \frac{f}{\lambda_0} \text{ pointwise, } \quad \varphi_{k_q} \rightarrow \varphi, \quad \text{and } \mu_{l_r} \rightarrow \mu \text{ weakly,}$$

by (3.4), we have

$$\lim_{p,q,r \rightarrow \infty} \varphi_{k_q}^{-1} \left(\int_{S^{n-1}} \varphi_{k_q} \left(\frac{f_{i_p}}{\|f_{i_p} : \mu_{l_r}\|_{\varphi_{k_q}}} \right) d\mu_{l_r} \right) = \varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{f}{\lambda_0} \right) d\mu \right).$$

Meanwhile, since

$$\varphi_{k_q}^{-1} \left(\int_{S^{n-1}} \varphi_{k_q} \left(\frac{f_{i_p}}{\|f_{i_p} : \mu_{l_r}\|_{\varphi_{k_q}}} \right) d\mu_{l_r} \right) = 1, \quad \text{for each } (p, q, r),$$

it yields that

$$\lim_{p,q,r \rightarrow \infty} \varphi_{k_q}^{-1} \left(\int_{S^{n-1}} \varphi_{k_q} \left(\frac{f_{i_p}}{\|f_{i_p} : \mu_{l_r}\|_{\varphi_{k_q}}} \right) d\mu_{l_r} \right) = 1.$$

Hence,

$$\varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{f}{\lambda_0} \right) d\mu \right) = 1.$$

From Lemma 2.1, it follows that $\lambda_0 = \|f : \mu\|_\varphi$.

The proof is complete. \square

Using Lemma 3.7, we immediately obtain

Lemma 3.8. *Suppose $K, K_i, L, L_j \in \mathcal{S}_o^n$ and $\varphi, \varphi_k \in \Phi$, $i, j, k \in \mathbb{N}$. If $K_i \rightarrow K$, $L_j \rightarrow L$ and $\varphi_k \rightarrow \varphi$, then*

$$\lim_{i,j,k \rightarrow \infty} \tilde{V}_{\varphi_k}(K_i, L_j) = \tilde{V}_\varphi(K, L),$$

$$\lim_{i,j,k \rightarrow \infty} \bar{\tilde{V}}_{\varphi_k}(K_i, L_j) = \bar{\tilde{V}}_\varphi(K, L),$$

and

$$\lim_{i,j,k \rightarrow \infty} O_{\varphi_k}(K_i, L_j) = O_\varphi(K, L).$$

Proof. That $K_i \rightarrow K$ and $L_j \rightarrow L$ yields ρ_{K_i}/ρ_{L_j} and ρ_K/ρ_L are strictly positive continuous on S^{n-1} ; $\rho_{K_i}/\rho_{L_j} \rightarrow \rho_K/\rho_L$; $\tilde{V}_{K_i} \rightarrow \tilde{V}_K$ weakly, and $V_{K_i}^* \rightarrow V_K^*$ weakly. Combining these facts and applying Lemma 3.7, the desired limits can be derived directly. \square

Recall that

$$\bar{\tilde{V}}_{-1}(K, L) = \int_{S^{n-1}} \frac{\rho_K}{\rho_L} dV_K^*, \quad \text{for } K, L \in \mathcal{S}_o^n.$$

The next lemma will be used in Section 6.

Lemma 3.9. *Suppose $K, L \in \mathcal{S}_o^n$, $\varphi \in \Phi$ and $p \in [1, \infty)$. Then*

(1) $\tilde{V}_{\varphi^p}(K, L)$ is increasing and bounded from above in p , and bounded from below by $\bar{\tilde{V}}_{-1}(K, L)$.

(2) $\lim_{p \rightarrow \infty} \tilde{V}_{\varphi^p}(K, L) = \left\| \frac{\rho_K}{\rho_L} \right\|_\infty$.

(3) $O_{\varphi^p}(K, L)$ is increasing and bounded from above in p , and bounded from below by $\bar{\tilde{V}}_{-1}(K, L)$.

(4) $\lim_{p \rightarrow \infty} O_{\varphi^p}(K, L) = \left\| \frac{\rho_K}{\rho_L} \right\|_\infty$.

Proof. Let $\lambda \in (0, \infty)$. From Definition 3.1, we have

$$\bar{\tilde{V}}_{\varphi^p}(K, \lambda L) = \varphi^{-1} \left(\left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right)^p dV_K^* \right)^{1/p} \right).$$

By Jensen's inequality, $\left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right)^p dV_K^* \right)^{1/p}$ is increasing in $p \in [1, \infty)$. Since φ^{-1} is also increasing in $(0, \infty)$, it yields that $\bar{\tilde{V}}_{\varphi^p}(K, \lambda L)$ is increasing in $p \in [1, \infty)$.

Since φ^{-1} and φ are both continuous and strictly increasing on $[0, \infty)$, it follows that

$$\begin{aligned}
\lim_{p \rightarrow \infty} \tilde{V}_{\varphi^p}(K, \lambda L) &= \lim_{p \rightarrow \infty} \varphi^{-1} \left(\left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right)^p dV_K^* \right)^{1/p} \right) \\
&= \varphi^{-1} \left(\lim_{p \rightarrow \infty} \left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right)^p dV_K^* \right)^{1/p} \right) \\
&= \varphi^{-1} \left(\max \left\{ \varphi \left(\frac{\rho_K(u)}{\lambda \rho_L(u)} \right) : u \in S^{n-1} \right\} \right) \\
&= \varphi^{-1} \left(\varphi \left(\max \left\{ \frac{\rho_K(u)}{\lambda \rho_L(u)} : u \in S^{n-1} \right\} \right) \right) \\
&= \frac{1}{\lambda} \left\| \frac{\rho_K}{\rho_L} \right\|_{\infty}.
\end{aligned}$$

Thus, $\tilde{V}_{\varphi^p}(K, \lambda L)$ is bounded from above by $\frac{1}{\lambda} \left\| \frac{\rho_K}{\rho_L} \right\|_{\infty}$.

From the definition of $\tilde{V}_{-1}(K, \lambda L)$, the strict monotonicity of φ^{-1} together with the convexity of φ and Jensen's inequality, and the definition of $\tilde{V}_{\varphi}(K, L)$, we have

$$\begin{aligned}
\tilde{V}_{-1}(K, \lambda L) &= \varphi^{-1} \left(\varphi \left(\int_{S^{n-1}} \frac{\rho_K}{\lambda \rho_L} dV_K^* \right) \right) \\
&\leq \varphi^{-1} \left(\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right) dV_K^* \right) \\
&= \tilde{V}_{\varphi}(K, \lambda L).
\end{aligned}$$

Thus, $\tilde{V}_{-1}(K, \lambda L) \leq \tilde{V}_{\varphi}(K, \lambda L)$.

Let $\lambda = 1$, it gives (1) and (2) directly.

Recall that

$$O_{\varphi^p}(K, L) = \inf \left\{ \lambda > 0 : \tilde{V}_{\varphi^p}(K, \lambda^{-1} L) \leq 1 \right\}.$$

So, for $1 \leq p < q < \infty$, from (1) we have

$$\tilde{V}_{-1}(K, \lambda^{-1} L) \leq \tilde{V}_{\varphi}(K, \lambda^{-1} L) \leq \tilde{V}_{\varphi^p}(K, \lambda^{-1} L) \leq \tilde{V}_{\varphi^q}(K, \lambda^{-1} L) \leq \lambda \left\| \frac{\rho_K}{\rho_L} \right\|_{\infty}.$$

Thus, we obtain

$$\tilde{V}_{-1}(K, L) \leq O_{\varphi}(K, L) \leq O_{\varphi^p}(K, L) \leq O_{\varphi^q}(K, L) \leq \left\| \frac{\rho_K}{\rho_L} \right\|_{\infty},$$

which implies (3) immediately.

By (3), any subsequence $\{O_{\varphi^{p_j}}(K, L)\}_j$, with $\lim_{j \rightarrow \infty} p_j = \infty$, must converge to certain number $\lambda_0 \in [\bar{V}_{-1}(K, L), \|\rho_K/\rho_L\|_\infty]$. So, to prove (4), it suffices to prove

$$\lambda_0 = \left\| \frac{\rho_K}{\rho_L} \right\|_\infty.$$

For brevity, let

$$\lambda_\infty = \left\| \frac{\rho_K}{\rho_L} \right\|_\infty \quad \text{and} \quad \lambda_j = O_{\varphi^{p_j}}(K, L).$$

For each j , define

$$g_j(\lambda) = \left[\int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\lambda \rho_L} \right)^{p_j} dV_K^* \right]^{1/p_j},$$

and

$$g_\infty(\lambda) = \varphi \left(\lambda^{-1} \left\| \frac{\rho_K}{\rho_L} \right\|_\infty \right).$$

Note that the functions g_j and g_∞ are continuous on $[\lambda_1, \lambda_\infty]$, and $g_j \rightarrow g_\infty$ pointwise on $[\lambda_1, \lambda_\infty]$ by (1). Thus, $g_j \rightarrow g_\infty$, uniformly on $[\lambda_1, \lambda_\infty]$.

Consequently, we have

$$\lim_{j \rightarrow \infty} g_j(\lambda_j) = \left(\lim_{j \rightarrow \infty} g_j \right) \left(\lim_{j \rightarrow \infty} \lambda_j \right) = g_\infty(\lambda_0).$$

Note that $g_j(\lambda_j) = \varphi(1)$ for each j . Hence, we obtain

$$g_\infty(\lambda_0) = \varphi(1); \quad \text{i.e.,} \quad \lambda_0 = \left\| \frac{\rho_K}{\rho_L} \right\|_\infty.$$

The proof is complete. □

4. Orlicz-Legendre ellipsoids

Let $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. For any $T \in \text{SL}(n)$, by Lemma 3.6 it gives

$$\bar{V}_\varphi(K, TB) \geq \left(\frac{V(K)}{\omega_n} \right)^{\frac{1}{n}} \quad \text{and} \quad O_\varphi(K, TB) \geq \left(\frac{V(K)}{\omega_n} \right)^{\frac{1}{n}}.$$

In view of the intimate connection between \bar{V}_φ and O_φ , to find the so-called Orlicz-Legendre ellipsoids, we also consider the following three problems, which are closely related to our originally posed Problem \tilde{S}_φ .

Problem P₁. *Find an ellipsoid E , amongst all origin-symmetric ellipsoids, which solves the constrained minimization problem*

$$\min \bar{V}_\varphi(K, E) \quad \text{subject to} \quad V(E) \leq \omega_n.$$

Problem P₂. Find an ellipsoid E , amongst all origin-symmetric ellipsoids, which solves the constrained minimization problem

$$\min O_\varphi(K, E) \quad \text{subject to} \quad V(E) \leq \omega_n.$$

The homogeneity of volume functional and Orlicz norm prompts us to consider the following Problem P₃, which is in some sense dual to Problem P₂.

Problem P₃. Find an ellipsoid E , amongst all origin-symmetric ellipsoids, which solves the constrained maximization problem

$$\max \left(\frac{\omega_n}{V(E)} \right)^{\frac{1}{n}} \quad \text{subject to} \quad O_\varphi(K, E) \leq 1.$$

In order to convenient comparison, we restate Problem \tilde{S}_φ as the following.

Problem \tilde{S}_φ . Find an ellipsoid E , amongst all origin-symmetric ellipsoids, which solves the constrained maximization problem

$$\max \left(\frac{\omega_n}{V(E)} \right)^{\frac{1}{n}} \quad \text{subject to} \quad \tilde{V}_\varphi(K, E) \leq 1.$$

Two observations are in order. First, from Definition 3.1 together with the fact that φ^{-1} is strictly increasing in $(0, \infty)$, the objective functional in P₁ can be replaced by $\tilde{V}_\varphi(K, E)$. Second, by the fact $V(E)V(E^*) = \omega_n^2$, the objective functional in P₃ and \tilde{S}_φ can be replaced by $V(E^*)$.

This section is organized as follows. After proving Lemmas 4.1 and 4.2, we prove Theorems 4.3 and 4.4, which demonstrate the existence and uniqueness of solution to P₁, respectively. The connection between P₁ and P₂ is established by Lemma 4.5, then the unique existence of solution to P₂ is shown in Theorem 4.6. Theorem 4.7 shows that the solutions to P₂ and P₃ only differ by a scale factor. Thus, the unique existence of solution to P₃ is confirmed. Lemma 4.8 reveals that P₃ and \tilde{S}_φ are essentially identical, so the proof of the unique existence of solution to \tilde{S}_φ is complete. Therefore, the notion of Orlicz-Legendre ellipsoid is ready to come out.

Lemma 4.1. Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then

$$\lim_{\substack{T \in \text{SL}(n) \\ \|T\| \rightarrow \infty}} \tilde{V}_\varphi(TK, B) = \infty,$$

and

$$\lim_{\substack{T \in \text{SL}(n) \\ \|T\| \rightarrow \infty}} O_\varphi(TK, B) = \infty.$$

Proof. Let $r_K = \min_{S^{n-1}} \rho_K$. Then $r_K B \subseteq K$. In addition, there exists a positive $r > 0$, say $r = \frac{1}{\sqrt{n}} r_K$, such that $r[-1, 1]^n \subseteq r_K B$. For $T \in \text{SL}(n)$, write T in the form $T = O_1 A O_2$,

where A is an $n \times n$ diagonal matrix, with $\det(A) = 1$ and positive diagonal elements a_1, \dots, a_n , and O_1, O_2 are $n \times n$ orthogonal matrices.

From the definition of the measure \tilde{V}_{AO_2K} , the polar coordinate formula, and the fact $K \supseteq r_K B$, the orthogonality of O_2 , the fact $r_K B \supseteq r[-1, 1]^n$, and finally the symmetry of Ax in x and $[-1, 1]^n$ with respect to o , we have

$$\begin{aligned}
\int_{S^{n-1}} \rho_{AO_2K} d\tilde{V}_{AO_2K} &= \frac{1}{n} \int_{S^{n-1}} \rho_{AO_2K}^{n+1} dS \\
&= \frac{n+1}{n} \int_K |AO_2x| dx \\
&\geq \frac{n+1}{n} \int_{r_K B} |AO_2x| dx \\
&= \frac{n+1}{n} \int_{r_K B} |Ax| dx \\
&\geq \frac{n+1}{n} \int_{r[-1, 1]^n} |Ax| dx \\
&= \frac{(n+1)2^n r^{n+1}}{n} \int_{[0, 1]^n} |Ax| dx.
\end{aligned}$$

For any $y \in \mathbb{R}^n$, let $\|y\|_1$ denote the l_1 norm of y . Recall that there exists a positive C such that $|y| \geq C\|y\|_1$, and $\sum_{i=1}^n a_i \geq \max_{1 \leq i \leq n} a_i = \|A\| = \|T\|$. So, we have

$$\int_{[0, 1]^n} |Ax| dx \geq \int_{[0, 1]^n} C\|Ax\|_1 dx \geq \frac{C}{2} \sum_{i=1}^n a_i \geq \frac{C}{2} \|T\|.$$

Thus, we obtain

$$(4.1) \quad \int_{S^{n-1}} \rho_{AO_2K} d\tilde{V}_{AO_2K} \geq \frac{(n+1)2^{n-1}r^{n+1}C}{n} \|T\|.$$

Now, from Definition 3.1 together with Lemma 3.2 (1), the convexity of φ together with Jensen's inequality, the strict monotonicity of φ together with (4.1), and the fact $V(AO_2K) = V(K)$, we have

$$\begin{aligned}
\frac{\tilde{V}_\varphi(TK, B)}{V(TK)} &= \frac{1}{V(AO_2K)} \int_{S^{n-1}} \varphi(\rho_{AO_2K}) d\tilde{V}_{AO_2K} \\
&\geq \varphi\left(\frac{1}{V(AO_2K)} \int_{S^{n-1}} \rho_{AO_2K} d\tilde{V}_{AO_2K}\right) \\
&\geq \varphi\left(\frac{(n+1)2^{n-1}r^{n+1}C}{nV(K)} \|T\|\right).
\end{aligned}$$

That is,

$$(4.2) \quad \frac{\tilde{V}_\varphi(TK, B)}{V(K)} \geq \varphi \left(\frac{2^{n-1}r^{n+1}(n+1)C}{nV(K)} \|T\| \right).$$

By the strict monotonicity of φ again, it immediately yields

$$\lim_{\substack{T \in \text{SL}(n) \\ \|T\| \rightarrow \infty}} \tilde{V}_\varphi(TK, B) = \infty.$$

Let

$$C_1 = \frac{2^{n-1}r^{n+1}(n+1)C}{nV(K)}.$$

Note that r depends on K . Applying (4.2) to the star body $O_\varphi(TK, B)^{-1}K$ and using Lemma 3.5, we obtain

$$\varphi(1) = \frac{\tilde{V}_\varphi(O_\varphi(TK, B)^{-1}TK, B)}{V(O_\varphi(TK, B)^{-1}K)} \geq \varphi(O_\varphi(TK, B)^{-1}C_1 \|T\|).$$

So, from the injectivity of φ , it follows that

$$O_\varphi(TK, B) \geq C_1 \|T\|,$$

which immediately yields

$$\lim_{\substack{T \in \text{SL}(n) \\ \|T\| \rightarrow \infty}} O_\varphi(TK, B) = \infty,$$

as desired. □

From Lemmas 4.1, 2.2, 3.2 and 3.4, we immediately obtain

Lemma 4.2. *Suppose $T \in \text{SL}(n)$. Then*

$$\lim_{\substack{T \in \text{SL}(n) \\ \|T\| \rightarrow \infty}} \tilde{V}_\varphi(K, TB) = \infty,$$

and

$$\lim_{\substack{T \in \text{SL}(n) \\ \|T\| \rightarrow \infty}} O_\varphi(K, TB) = \infty.$$

Now, using Lemmas 4.2 and 2.3, we can prove the existence of solution to problem P_1 .

Theorem 4.3. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then there exists an solution to P_1 .*

Proof. First, we prove that any $E \in \mathcal{E}^n$ with $V(E) < \omega_n$ cannot be a solution to P_1 .

Indeed, let $\lambda_0 = (\omega_n/V(E))^{1/n}$, then $\lambda_0 E$ also satisfies the constraint condition in P_1 . From the fact that φ is strictly increasing on $[0, \infty)$ together with Definition 3.1, it necessarily results in that $\tilde{V}_\varphi(K, \lambda_0 E) < \tilde{V}_\varphi(K, E)$.

Hence, Problem P_1 can be equivalently restated as

$$\inf \left\{ \tilde{V}_\varphi(K, TB) : T \in \text{SL}(n) \right\}.$$

Observe that the infimum exists, since

$$V(K)\varphi \left(\left(\frac{V(K)}{\omega_n} \right)^{\frac{1}{n}} \right) \leq \inf \left\{ \tilde{V}_\varphi(K, TB) : T \in \text{SL}(n) \right\} \leq \tilde{V}_\varphi(K, B) < \infty,$$

where the left inequality follows from Lemma 3.6 and Definition 3.1.

Let

$$\mathcal{T} = \left\{ T \in \text{SL}(n) : \tilde{V}_\varphi(K, TB) \leq \tilde{V}_\varphi(K, B) \right\}.$$

From Lemma 2.2 (3) and Lemma 3.8, $\tilde{V}_\varphi(K, TB)$ is continuous in $T \in (\text{SL}(n), d_n)$. Thus, the set \mathcal{T} is closed in $(\text{SL}(n), d_n)$. Meanwhile, the definition of \mathcal{T} and Lemma 4.2 guarantee that \mathcal{T} is bounded in $(\text{SL}(n), d_n)$. Hence, \mathcal{T} is compact.

Now, since $\tilde{V}_\varphi(K, TB)$ is continuous on (\mathcal{T}, d_n) , it concludes that there exists a $T_0 \in \mathcal{T}$ such that

$$\tilde{V}_\varphi(K, T_0 B) = \min \{ \tilde{V}_\varphi(K, TB) : T \in \mathcal{T} \} = \inf \{ \tilde{V}_\varphi(K, TB) : T \in \text{SL}(n) \},$$

which completes the proof. \square

Theorem 4.4. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then, modulo orthogonal transformations, there exists a unique $\text{SL}(n)$ transformation solving the extremal problem*

$$\min \left\{ \tilde{V}_\varphi(K, TB) : T \in \text{SL}(n) \right\}.$$

Equivalently, there exists a unique solution to Problem P_1 .

Proof. The existence is shown by Theorem 4.3. We only need to prove the uniqueness. For this aim, we argue by contradiction.

Assume that $T_1, T_2 \in \text{SL}(n)$ both solve the considered minimization problem. Let $E_1 = T_1 B$, $E_2 = T_2 B$. It is known that each $T \in \text{SL}(n)$ can be represented in the form $T = PQ$, where P is symmetric, positive definite and Q is orthogonal. So, w.l.o.g., we may assume that T_1, T_2 are symmetric and positive definite.

By the Minkowski inequality for symmetric and positive definite matrices, we have

$$\det \left(\frac{T_1^{-1} + T_2^{-1}}{2} \right)^{\frac{1}{n}} > \frac{1}{2} \det(T_1^{-1})^{\frac{1}{n}} + \frac{1}{2} \det(T_2^{-1})^{\frac{1}{n}} = 1.$$

Let

$$T_3^{-1} = \det \left(\frac{T_1^{-1} + T_2^{-1}}{2} \right)^{-\frac{1}{n}} \frac{T_1^{-1} + T_2^{-1}}{2}.$$

Then $T_3 \in \text{SL}(n)$ is symmetric.

Let $E_3 = T_3 B$. For all $u \in S^{n-1}$, we have

$$\begin{aligned} h_{E_3^*}(u) &= h_{T_3^{-1}B}(u) \\ &< h_{\frac{T_1^{-1}+T_2^{-1}}{2}B}(u) = \left| \frac{T_1^{-1}u + T_2^{-1}u}{2} \right| \\ &\leq \frac{|T_1^{-1}u| + |T_2^{-1}u|}{2} = \frac{1}{2}h_{T_1^{-1}B} + \frac{1}{2}h_{T_2^{-1}B}. \end{aligned}$$

Since $E_i^* = T_i^{-1}B$, $i = 1, 2, 3$, it follows that

$$\tilde{V}_\varphi(K, E_i) = \int_{S^{n-1}} \varphi \left(\rho_K h_{T_i^{-1}B} \right) d\tilde{V}_K.$$

From the fact that φ is strictly increasing and convex in $[0, \infty)$, we have

$$\varphi \left(\rho_K h_{T_3^{-1}B} \right) < \frac{1}{2} \varphi \left(\rho_K h_{T_1^{-1}B} \right) + \frac{1}{2} \varphi \left(\rho_K h_{T_2^{-1}B} \right).$$

Thus,

$$\tilde{V}_\varphi(K, E_3) < \frac{1}{2} \tilde{V}_\varphi(K, E_1) + \frac{1}{2} \tilde{V}_\varphi(K, E_2).$$

Hence,

$$\tilde{V}_\varphi(K, E_3) < \tilde{V}_\varphi(K, E_1) = \tilde{V}_\varphi(K, E_2).$$

However, from $T_3 \in \text{SL}(n)$ and the assumption on E_1 and E_2 , we also have

$$\tilde{V}_\varphi(K, E_3) \geq \tilde{V}_\varphi(K, E_1) = \tilde{V}_\varphi(K, E_2),$$

which contradicts the above. This completes the proof. \square

Lemma 4.5. Suppose $E_0 \in \mathcal{E}^n$ and $V(E_0) = \omega_n$. Then, for any $T \in \text{SL}(n)$,

$$\tilde{V}_\varphi(K, O_\varphi(K, E_0)E_0) \leq \tilde{V}_\varphi(K, O_\varphi(K, E_0)TE_0)$$

if and only if

$$O_\varphi(K, E_0) \leq O_\varphi(K, TE_0).$$

Proof. From Definition 3.1 together with the strict monotonicity of φ^{-1} , Lemma 3.5, and Lemma 2.1 together with Definition 3.3, it follows that

$$\begin{aligned} \tilde{V}_\varphi(K, O_\varphi(K, E_0)E_0) &\leq \tilde{V}_\varphi(K, O_\varphi(K, E_0)TE_0) \\ \iff \bar{\tilde{V}}_\varphi(K, O_\varphi(K, E_0)E_0) &\leq \bar{\tilde{V}}_\varphi(K, O_\varphi(K, E_0)TE_0) \\ \iff 1 \leq \bar{\tilde{V}}_\varphi(K, O_\varphi(K, E_0)TE_0) & \\ \iff \bar{\tilde{V}}_\varphi(K, O_\varphi(K, TE_0)TE_0) &\leq \bar{\tilde{V}}_\varphi(K, O_\varphi(K, E_0)TE_0) \\ \iff \tilde{V}_\varphi(K, O_\varphi(K, TE_0)TE_0) &\leq \tilde{V}_\varphi(K, O_\varphi(K, E_0)TE_0) \\ \iff O_\varphi(K, E_0) &\leq O_\varphi(K, TE_0), \end{aligned}$$

as desired. □

From Theorem 4.4 and Lemma 4.5, we can prove the following.

Theorem 4.6. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then there exists a unique solution to Problem P_2 .*

Proof. First, we prove the existence of solution to problem P_2 . Observe that the constraint condition in P_2 can be turned into $V(E) = \omega_n$. Indeed, for any $s \in (0, 1)$ and $E \in \mathcal{E}^n$ with $V(E) = \omega_n$, by Lemma 3.4 it follows that

$$O_\varphi(K, sE) = s^{-1}O_\varphi(K, E) > O_\varphi(K, E),$$

which indicates that sE cannot be a solution to P_2 .

Let $\lambda_0 = \inf \{O_\varphi(K, TB) : T \in \text{SL}(n)\}$. From Lemma 3.6, we have

$$0 < \left(\frac{V(K)}{\omega_n} \right)^{\frac{1}{n}} \leq \lambda_0 \leq O_\varphi(K, B) < \infty.$$

Similar to the proof of Theorem 4.3, we can show the set

$$\{T \in \text{SL}(n) : O_\varphi(K, TB) \leq O_\varphi(K, B)\}$$

is also compact. Combining it with the continuity of $O_\varphi(K, TB)$, the existence of solution to P_2 is demonstrated.

Now, we proceed to prove the uniqueness.

Assume ellipsoid E_0 is a solution to P_2 . Then

$$O_\varphi(K, E_0) \leq O_\varphi(K, TE_0), \quad \text{for } T \in \text{SL}(n).$$

By Lemma 4.5, it follows that

$$\tilde{V}_\varphi(K, O_\varphi(K, E_0)E_0) \leq \tilde{V}_\varphi(K, O_\varphi(K, E_0)TE_0), \quad \text{for } T \in \text{SL}(n).$$

Thus, E_0 is a solution to Problem P_1 for star body $\lambda_0^{-1}K$. Hence, by Theorem 4.4, the solution to P_2 is unique. □

Theorem 4.7. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

(1) *If E_0 is the unique solution to Problem P_2 , then $O_\varphi(K, E_0)E_0$ is a solution to Problem P_3 .*

(2) *If E_1 is a solution to Problem P_3 , then $\left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1$ is a solution to Problem P_2 .*

Consequently, there exists a unique solution to Problem P_3 .

Proof. (1) Let $E \in \mathcal{E}^n$ with $O_\varphi(K, E) \leq 1$. Since $\left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{n}} E$ satisfies the constraint condition of P_2 , by Lemma 3.4 (2), the fact $V(E_0) = \omega_n$, and the assumption $O_\varphi(K, E) \leq 1$, we have

$$\begin{aligned}
V(O_\varphi(K, E_0)E_0) &= O_\varphi(K, E_0)^n V(E_0) \\
&\leq O_\varphi\left(K, \left(\frac{\omega_n}{V(E)}\right)^{\frac{1}{n}} E\right)^n V(E_0) \\
&= \frac{V(E)}{\omega_n} O_\varphi(K, E)^n V(E_0) \\
&= V(E) O_\varphi(K, E)^n \\
&\leq V(E).
\end{aligned}$$

Thus,

$$V(O_\varphi(K, E_0)E_0) \leq V(E); \quad \text{i.e.,} \quad \frac{\omega_n}{V(O_\varphi(K, E_0)E_0)} \geq \frac{\omega_n}{V(E)},$$

which shows that $O_\varphi(K, E_0)E_0$ solves Problem P_3 .

(2) First, we prove that the constraint condition in P_3 can be turned into $O_\varphi(K, E) = 1$; i.e., a solution E_1 to P_3 must satisfies $O_\varphi(K, E_1) = 1$.

Indeed, let $E \in \mathcal{E}^n$ with $O_\varphi(K, E) < 1$. By Lemma 3.4 (2), $O_\varphi(K, O_\varphi(K, E)E) = 1$. Since

$$\frac{\omega_n}{V(O_\varphi(K, E)E)} = \frac{\omega_n}{O_\varphi(K, E)^n V(E)} > \frac{\omega_n}{V(E)},$$

it implies that E cannot be a solution to Problem P_3 .

Now, we can finish the proof of (2).

Let $E' \in \mathcal{E}^n$ with $V(E') \leq \omega_n$. By Lemma 3.4 (2), $O_\varphi(K, O_\varphi(K, E')E') = 1$. Thus $O_\varphi(K, E')E'$ satisfies the constraint condition of Problem P_3 . Since E_1 is a solution to Problem P_3 , it follows that

$$V(O_\varphi(K, E')E') \geq V(E_1).$$

So, by the assumption $V(E') \leq \omega_n$, the fact $O_\varphi(K, E_1) = 1$ and Lemma 3.4 (2), we have

$$\begin{aligned}
O_\varphi(K, E') &\geq \left(\frac{V(E_1)}{V(E')}\right)^{\frac{1}{n}} \geq \left(\frac{V(E_1)}{\omega_n}\right)^{\frac{1}{n}} \\
&= O_\varphi(K, E_1) \left(\frac{V(E_1)}{\omega_n}\right)^{\frac{1}{n}} \\
&= O_\varphi\left(K, \left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1\right).
\end{aligned}$$

Thus,

$$O_\varphi(K, E') \geq O_\varphi\left(K, \left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1\right),$$

which shows that $\left(\frac{\omega_n}{V(E_1)}\right)^{\frac{1}{n}} E_1$ solves Problem P₂. \square

Lemma 4.8. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then*

- (1) $\min_{\{E \in \mathcal{E}^n : O_\varphi(K, E) \leq 1\}} V(E) = \min_{\{E \in \mathcal{E}^n : O_\varphi(K, E) = 1\}} V(E).$
- (2) $\{E \in \mathcal{E}^n : O_\varphi(K, E) = 1\} = \left\{E \in \mathcal{E}^n : \tilde{V}_\varphi(K, E) = 1\right\}.$
- (3) $\min_{\{E \in \mathcal{E}^n : \tilde{V}_\varphi(K, E) = 1\}} V(E) = \min_{\{E \in \mathcal{E}^n : \tilde{V}_\varphi(K, E) \leq 1\}} V(E).$

Consequently, the solutions to Problems P₃ and \tilde{S}_φ are identical.

Proof. The proof of (1) can be referred to the proof of Theorem 4.7 (2). Assertion (2) follows from Lemma 3.5 directly.

Now, we prove assertion (3). Let $E \in \mathcal{E}^n$ with $\tilde{V}_\varphi(K, E) < 1$. From Definition 3.1 and Lemma 2.1, we know that the unique positive λ_0 satisfying the equation

$$\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\lambda_0 \rho_E}\right) dV_K^* = \varphi(1)$$

is necessarily in $(0, 1)$, and

$$\int_{S^{n-1}} \varphi\left(\frac{\rho_K}{\lambda' \rho_E}\right) dV_K^* \leq \varphi(1), \quad \text{i.e.,} \quad \tilde{V}_\varphi(K, \lambda' E) < 1, \quad \text{for any } \lambda' \in (\lambda_0, 1).$$

At the same time, since $V(\lambda' E) < V(E)$, $\forall \lambda' \in (\lambda_0, 1)$, so E cannot possibly solve the minimization problem

$$\min \left\{ V(E) : E \in \mathcal{E}^n \text{ and } \tilde{V}_\varphi(K, E) \leq 1 \right\}.$$

Hence, assertion (3) is derived.

From the proved (1), (2) and (3), we can conclude that Problem P₃ and Problem \tilde{S}_φ have the same solution. \square

For different dilations $\lambda_1 K$ and $\lambda_2 K$, $\lambda_1, \lambda_2 > 0$, Problems P₁ do not generally have the identical solution. By contrast, the homogeneity of $O_\varphi(\lambda K, L)$ in $\lambda \in (0, \infty)$ guarantees that all Problems P₂ for λK in $\lambda \in (0, \infty)$ have the identical unique solution. Problems P₃ and \tilde{S}_φ are identical, and Problem P₃ is the dual problem of P₂. Thus, Problem \tilde{S}_φ is not the dual problem of P₁ in general.

In view of Theorem 4.6, Theorem 4.7 and Lemma 4.8, we are in the position to introduce a family of ellipsoids in the framework of dual Orlicz Brunn-Minkowski theory, which are extensions of Legendre ellipsoid.

Definition 4.9. Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Amongst all origin-symmetric ellipsoids E , the unique ellipsoid that solves the constrained minimization problem

$$\min_E V(E) \quad \text{subject to} \quad O_\varphi(K, E) \leq 1$$

is called the *Orlicz-Legendre ellipsoid* of K with respect to φ , and is denoted by $L_\varphi K$.

Amongst all origin-symmetric ellipsoids E , the unique ellipsoid that solves the constrained minimization problem

$$\min_E O_\varphi(K, E) \quad \text{subject to} \quad V(E) = \omega_n$$

is called the *normalized Orlicz-Legendre ellipsoid* of K with respect to φ , and is denoted by $\bar{L}_\varphi K$.

For the polar of $L_\varphi K$ or $\bar{L}_\varphi K$, we write $L_\varphi^* K$ or $\bar{L}_\varphi^* K$, rather than $(L_\varphi K)^*$ or $(\bar{L}_\varphi K)^*$.

If $\varphi(t) = t^p$, $1 \leq p < \infty$, we write $L_\varphi K$ and $\bar{L}_\varphi K$ for $L_p K$ and $\bar{L}_p K$, respectively. Especially, $L_2 K$ is precisely the Legendre ellipsoid $\Gamma_2 K$.

We observe that for the case $\varphi(t) = t^p$, Problems P_1 and P_2 are identical, and were previously solved by Bastero and Romance [5]. Based on their works, Yu [64] introduced the ellipsoids $L_p K$ for convex bodies containing the origin in their interiors.

From Theorem 4.7, it is obvious that

$$(4.3) \quad L_\varphi K = O_\varphi(K, \bar{L}_\varphi K) \bar{L}_\varphi K \quad \text{and} \quad \bar{L}_\varphi K = \left(\frac{\omega_n}{V(L_\varphi K)} \right)^{\frac{1}{n}} L_\varphi K.$$

Definition 4.9 combined with inequality (3.1) shows that for any $E \in \mathcal{E}^n$,

$$L_\varphi E = E.$$

From Definition 4.9 and Lemma 3.4, we easily know that the operator L_φ intertwines with elements of $GL(n)$.

Lemma 4.10. Suppose $K \in \mathcal{S}^n$ and $\varphi \in \Phi$. Then for any $T \in GL(n)$,

$$L_\varphi(TK) = T(L_\varphi K).$$

Incidentally, we introduce the following.

Definition 4.11. Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Amongst all origin-symmetric ellipsoids E , the unique ellipsoid which solves the constrained minimization problem

$$\min_E \tilde{V}_\varphi(K, E) \quad \text{subject to} \quad V(E) = \omega_n$$

is denoted by $L_\varphi^\diamond K$.

Obviously, if $\varphi(t) = t^p$, $1 \leq p < \infty$, then $L_\varphi^\diamond K = \bar{L}_p K$.

5. The continuity of Orlicz-Legendre ellipsoids

In this section, we aim to show the continuity of Orlicz-Legendre ellipsoids $L_\varphi K$ with respect to φ and K .

Throughout this section, we suppose $\varphi \in \Phi$, $K, K_i \in \mathcal{S}_o^n$, $\varphi, \varphi_j \in \Phi$, $i, j \in \mathbb{N}$, and $K_i \rightarrow K$ and $\varphi_j \rightarrow \varphi$. It is easily seen that there exist positive r_m and r_M , such that

$$r_m B \subseteq K \subseteq r_M B \quad \text{and} \quad r_m B \subseteq K_i \subseteq r_M B \quad \text{for each } i \in \mathbb{N}.$$

Lemma 5.1. $\sup \left\{ d(L_\varphi^* K), d(L_\varphi^* K_i), d(L_{\varphi_j}^* K), d(L_{\varphi_j}^* K_i), \text{ with } i, j \in \mathbb{N} \right\} < \infty$.

Proof. Let $E \in \mathcal{E}^n$. First, we prove the implication

$$(5.1) \quad O_\varphi(K, E) \leq 1 \implies d(E^*) \leq \frac{n\omega_n}{2r_m\omega_{n-1}} \varphi^{-1} \left(\left(\frac{r_M}{r_m} \right)^n \varphi(1) \right).$$

Assume $O_\varphi(K, E) \leq 1$. From the definition of $O_\varphi(K, E)$ together with Lemma 2.1 and Lemma 3.5, the definition of $\tilde{V}_\varphi(K, E)$, the fact $r_m B \subseteq K \subseteq r_M B$ together with the monotonicity of φ , the convexity of φ together with Jensen's inequality, (2.3), the fact $h_{E^*}(u) \geq d(E^*)|v_{E^*} \cdot u|$ for $u \in S^{n-1}$, and finally Cauchy's projection formula, it follows that

$$\begin{aligned} \varphi(1) &\geq \frac{\tilde{V}_\varphi(K, E)}{V(K)} \\ &= \frac{1}{nV(K)} \int_{S^{n-1}} \varphi \left(\frac{\rho_K}{\rho_E} \right) \rho_K^n dS \\ &\geq \left(\frac{r_m}{r_M} \right)^n \frac{1}{n\omega_n} \int_{S^{n-1}} \varphi \left(\frac{r_m}{\rho_E} \right) dS \\ &\geq \left(\frac{r_m}{r_M} \right)^n \varphi \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \frac{r_m}{\rho_E} dS \right) \\ &= \left(\frac{r_m}{r_M} \right)^n \varphi \left(\frac{r_m}{n\omega_n} \int_{S^{n-1}} h_{E^*} dS \right) \\ &\geq \left(\frac{r_m}{r_M} \right)^n \varphi \left(\frac{r_m}{n\omega_n} \int_{S^{n-1}} d(E^*)|v_{E^*} \cdot u| dS(u) \right) \\ &= \left(\frac{r_m}{r_M} \right)^n \varphi \left(\frac{2r_m\omega_{n-1}}{n\omega_n} d(E^*) \right). \end{aligned}$$

Thus,

$$\varphi(1) \geq \left(\frac{r_m}{r_M} \right)^n \varphi \left(\frac{2r_m\omega_{n-1}}{n\omega_n} d(E^*) \right).$$

From the monotonicity of φ , it yields the inequality in (5.1).

Since that $\varphi_j \rightarrow \varphi$ implies $\varphi_j(1) \rightarrow \varphi(1)$ and $\varphi_j^{-1} \rightarrow \varphi^{-1}$, it follows that

$$\varphi_j^{-1} \left(\left(\frac{r_M}{r_m} \right)^n \varphi_j(1) \right) \rightarrow \varphi^{-1} \left(\left(\frac{r_M}{r_m} \right)^n \varphi(1) \right),$$

and therefore

$$\sup \left\{ \varphi^{-1} \left(\left(\frac{r_M}{r_m} \right)^n \varphi(1) \right), \varphi_j^{-1} \left(\left(\frac{r_M}{r_m} \right)^n \varphi_j(1) \right), \text{ with } j \in \mathbb{N} \right\} < \infty.$$

This, as well as (5.1), proves the desired lemma. \square

In light of Lemma 5.1 and Lemma 2.2, we can show

Lemma 5.2. $\sup \{ d(\bar{L}_\varphi K), d(\bar{L}_\varphi K_i), d(\bar{L}_{\varphi_j} K), d(\bar{L}_{\varphi_j} K_i), \text{ with } i, j \in \mathbb{N} \} < \infty.$

Proof. From (4.3), we have

$$\begin{aligned} d(\bar{L}_\varphi^* K) &= d \left(\left(\left(\frac{\omega_n}{V(L_\varphi K)} \right)^{\frac{1}{n}} L_\varphi K \right)^* \right) \\ &= \left(\frac{V(L_\varphi K)}{\omega_n} \right)^{\frac{1}{n}} d(L_\varphi^* K) \\ &\leq \left(\frac{V(L_\infty K)}{\omega_n} \right)^{\frac{1}{n}} d(L_\varphi^* K) \\ &\leq r_M d(L_\varphi^* K), \end{aligned}$$

That is,

$$(5.2) \quad d(\bar{L}_\varphi^* K) \leq r_M d(L_\varphi^* K).$$

Note that Definition 6.1, Theorem 6.2 and the inequality $V(L_\varphi K) \leq V(L_\infty K)$ given by Theorem 8.2, are previously used here.

Observe that (5.2) also holds when φ is replaced by φ_j or K is replaced by K_i . Thus, by Lemma 5.1, it follows

$$\sup \left\{ d(\bar{L}_\varphi^* K), d(\bar{L}_\varphi^* K_i), d(\bar{L}_{\varphi_j}^* K), d(\bar{L}_{\varphi_j}^* K_i), \text{ with } i, j \in \mathbb{N} \right\} < \infty.$$

Take $T_{(0,0)}, T_{(i,j)}, T_{(i,0)}, T_{(0,j)} \in \text{SL}(n)$, with $i, j \in \mathbb{N}$, such that

$$T_{(0,0)} B = \bar{L}_\varphi K, \quad T_{(i,0)} B = \bar{L}_\varphi K_i, \quad T_{(0,j)} B = \bar{L}_{\varphi_j} K, \quad T_{(i,j)} B = \bar{L}_{\varphi_j} K_i.$$

Then,

$$\sup \left\{ \|T_{(0,0)}^{-1}\|, \|T_{(i,j)}^{-1}\|, \|T_{(i,0)}^{-1}\|, \|T_{(0,j)}^{-1}\|, \text{ with } i, j \in \mathbb{N} \right\} < \infty.$$

This, together with Lemma 2.2, gives

$$\sup \left\{ \|T_{(0,0)}\|, \|T_{(i,j)}\|, \|T_{(i,0)}\|, \|T_{(0,j)}\|, \text{ with } i, j \in \mathbb{N} \right\} < \infty.$$

Hence, the desired lemma is proved. \square

Now, from Lemma 5.2, there exists a constant $R \in (0, \infty)$, such that all the ellipsoids $\bar{L}_\varphi K$, $\bar{L}_{\varphi_j} K$, $\bar{L}_\varphi K_i$ and $\bar{L}_{\varphi_j} K_i$ are in the set

$$\mathcal{E}_R = \{E \in \mathcal{E}^n : V(E) = \omega_n \text{ and } E \subseteq RB\}.$$

From the compactness of the sets \mathcal{E}_R and $\{K \in \mathcal{S}_o^n : r_m B \subseteq K \subseteq r_M K\}$, together with Lemma 3.8, we immediately obtain:

Lemma 5.3. *The limit $\lim_{i,j \rightarrow \infty} O_{\varphi_j}(K_i, E) = O_\varphi(K, E)$ is uniform in $E \in \mathcal{E}_R$.*

Lemma 5.4. $\lim_{i,j \rightarrow \infty} O_{\varphi_j}(K_i, \bar{L}_{\varphi_j} K_i) = O_\varphi(K, \bar{L}_\varphi K).$

Proof. From Definition 4.9 and Lemma 5.3, we have

$$\begin{aligned} \lim_{i,j \rightarrow \infty} O_{\varphi_j}(K_i, \bar{L}_{\varphi_j} K_i) &= \lim_{i,j \rightarrow \infty} \min_{E \in \mathcal{E}_R} O_{\varphi_j}(K_i, E) \\ &= \min_{E \in \mathcal{E}_R} \lim_{i,j \rightarrow \infty} O_{\varphi_j}(K_i, E) \\ &= \min_{E \in \mathcal{E}_R} O_\varphi(K, E) \\ &= O_\varphi(K, \bar{L}_\varphi K), \end{aligned}$$

as desired. □

Lemma 5.5. $\lim_{i,j \rightarrow \infty} \bar{L}_{\varphi_j} K_i = \bar{L}_\varphi K.$

Proof. We argue by contradiction and assume the proposition is false.

Then, from the compactness of \mathcal{E}_R and Lemma 2.3, there exists a convergent subsequence $\{\bar{L}_{\varphi_{j_q}} K_{i_p}\}_{p,q \in \mathbb{N}}$, such that

$$(5.3) \quad \lim_{i,j \rightarrow \infty} \bar{L}_{\varphi_j} K_i = E_0 \in \mathcal{E}_R \quad \text{and} \quad E_0 \neq \bar{L}_\varphi K.$$

From Lemma 3.8 and Lemma 5.4, it follows that

$$\begin{aligned} O_\varphi(K, \lim_{p,q \rightarrow \infty} \bar{L}_{\varphi_{j_q}} K_{i_p}) &= \lim_{p,q \rightarrow \infty} O_\varphi(K, \bar{L}_{\varphi_{j_q}} K_{i_p}) \\ &= \lim_{p,q \rightarrow \infty} \lim_{k \rightarrow \infty} O_{\varphi_k}(K, \bar{L}_{\varphi_{j_q}} K_{i_p}) \\ &= \lim_{p,q,k \rightarrow \infty} O_{\varphi_k}(K, \bar{L}_{\varphi_{j_q}} K_{i_p}) \\ &= \lim_{p,q \rightarrow \infty} O_{\varphi_{j_q}}(K, \bar{L}_{\varphi_{j_q}} K_{i_p}) \\ &= O_\varphi(K, \bar{L}_\varphi K). \end{aligned}$$

Since the solution to Problem P₂ is unique, we have

$$\lim_{p,q \rightarrow \infty} \bar{L}_{\varphi_{j_q}} K_{i_p} = \bar{L}_\varphi K,$$

which contradicts (5.3). □

Theorem 5.6. *Suppose $K, K_i \in \mathcal{S}_o^n$ and $\varphi, \varphi_j \in \Phi$, $i, j \in \mathbb{N}$. If $K_i \rightarrow K$ and $\varphi_j \rightarrow \varphi$, then*

$$\lim_{i,j \rightarrow \infty} L_{\varphi_j} K_i = L_{\varphi} K.$$

Proof. From Lemma 5.4, Lemma 5.5, together with the identity

$$L_{\varphi} K = O_{\varphi}(K, \bar{L}_{\varphi} K) \bar{L}_{\varphi} K,$$

the desired limit is immediately derived. \square

From Theorem 5.6, several corollaries are derived directly.

Corollary 5.7. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi, \varphi_j \in \Phi$, $j \in \mathbb{N}$. If $\varphi_j \rightarrow \varphi$, then*

$$\lim_{j \rightarrow \infty} L_{\varphi_j} K = L_{\varphi} K.$$

Corollary 5.8. *Suppose $K, K_i \in \mathcal{S}_o^n$ and $\varphi \in \Phi$, $i \in \mathbb{N}$. If $\varphi_i \rightarrow \varphi$, then*

$$\lim_{i \rightarrow \infty} L_{\varphi} K_i = L_{\varphi} K.$$

Corollary 5.9. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then $L_{\varphi^p} K$ is continuous in $p \in [1, \infty)$.*

Corollary 5.10. *The L_p Legendre ellipsoid $L_p K$ is continuous in $(K, p) \in \mathcal{S}_o^n \times [1, \infty)$.*

We observe that although Yu et.al [64] firstly introduced the notion of L_p Legendre ellipsoids, they did not consider the above continuity at all.

6. A common limit position

As Corollary 5.9 claims, for any $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$, the Orlicz-Legendre ellipsoid $L_{\varphi^p} K$ is continuous in $p \in [0, \infty)$. In this section, we show that as $p \rightarrow \infty$, $L_{\varphi^p} K$ approaches to a new ellipsoid $L_{\infty} K$, which is defined by the following.

Definition 6.1. For $K \in \mathcal{S}_o^n$, the ellipsoid $L_{\infty} K$ is defined by

$$L_{\infty} K = (E_{\infty}(\text{conv} K)^*)^*.$$

Here, $\text{conv} K$ denotes the convex hull of K . Write $\bar{L}_{\infty} K$ for its normalization, i.e.,

$$\bar{L}_{\infty} K = \left(\frac{\omega_n}{V(L_{\infty} K)} \right)^{\frac{1}{n}} L_{\infty} K.$$

The following two theorems show a fundamental feature of $L_{\infty} K$ and $\bar{L}_{\infty} K$.

Theorem 6.2. *Suppose $K \in \mathcal{S}_o^n$. Amongst all origin-symmetric ellipsoids that contain K , the ellipsoid $L_{\infty} K$ is the unique one with minimal volume.*

For a convex body $K \in \mathcal{K}_o^n$, if the John point of K^* is at the origin, then $(L_\infty K)^*$ is precisely the John ellipsoid $J(K^*)$ of K^* . If K is an origin-symmetric star body in \mathbb{R}^n , then $L_\infty K$ is precisely the Löwner ellipsoid of K .

Proof. First, observe that for $E \in \mathcal{E}^n$,

$$K \subseteq E \iff \text{conv} K \subseteq E.$$

Indeed, if $K \subseteq E$, then the fact $\text{conv} E = E$ yields the inclusion $\text{conv} K \subseteq E$; conversely, if $\text{conv} K \subseteq E$, then the fact $K \subseteq \text{conv} K$ yields the inclusion $K \subseteq E$.

Note that $\text{conv} K \in \mathcal{K}_o^n$. So, for $E \in \mathcal{E}^n$, it holds

$$\text{conv} K \subseteq E \iff E^* \subseteq (\text{conv} K)^*.$$

From this equivalence and the fact $V(E)V(E^*) = \omega_n^2$, we can reformulate the extremal problem

$$\min \{V(E) : E \in \mathcal{E}^n \text{ and } \text{conv} K \subseteq E\}$$

equivalently as

$$\max \{V(E^*) : E \in \mathcal{E}^n \text{ and } E^* \subseteq (\text{conv} K)^*\}.$$

Recall that the John ellipsoid $E_\infty(\text{conv} K)^*$ [48, 68] is the unique solution to the above maximization problem. Since $E_\infty(\text{conv} K)^*$ is the unique origin-symmetric ellipsoid of maximal volume contained in the convex body $(\text{conv} K)^*$, we know that $(E_\infty(\text{conv} K)^*)^*$ is the unique ellipsoid of minimal volume containing $\text{conv} K$. \square

Theorem 6.3. *Suppose $K \in \mathcal{S}_o^n$. Amongst all origin-symmetric ellipsoids E , the ellipsoid $\overline{L}_\infty K$ uniquely solves the constrained minimization problem*

$$\min_E \left\| \frac{\rho_K}{\rho_E} \right\|_\infty \quad \text{subject to} \quad V(E) \leq \omega_n.$$

Proof. The proof will be complete after two steps.

First, we show that the ellipsoid $\overline{L}_\infty K$ solves the desired extremal problem.

Let $E \in \mathcal{E}^n$ with $V(E) \leq \omega_n$. From the identity $\|\rho_K/\rho_{\|\rho_K/\rho_E\|_\infty E}\|_\infty = 1$ and the implication

$$\left\| \frac{\rho_K}{\rho_L} \right\|_\infty = 1 \implies L \supseteq K, \quad \text{for } L \in \mathcal{S}_o^n,$$

it follows that $\|\rho_K/\rho_E\|_\infty E \supseteq K$. Thus, by Theorem 6.2,

$$V(\|\rho_K/\rho_E\|_\infty E) \geq V(L_\infty K).$$

From this inequality, the assumption that $V(E) \leq \omega_n$, the fact that $\|\rho_K/\rho_{L_\infty K}\|_\infty = 1$, and finally the definition of $\bar{L}_\infty K$, it follows that

$$\begin{aligned}
\left\| \frac{\rho_K}{\rho_E} \right\|_\infty &\geq \left(\frac{V(L_\infty K)}{V(E)} \right)^{\frac{1}{n}} \\
&\geq \left(\frac{V(L_\infty K)}{\omega_n} \right)^{\frac{1}{n}} \\
&= \left\| \frac{\rho_K}{\rho_{L_\infty K}} \right\|_\infty \left(\frac{V(L_\infty K)}{\omega_n} \right)^{\frac{1}{n}} \\
&= \left\| \frac{\rho_K}{\rho_{\left(\frac{\omega_n}{V(L_\infty K)}\right)^{\frac{1}{n}} L_\infty K}} \right\|_\infty \\
&= \left\| \frac{\rho_K}{\rho_{\bar{L}_\infty K}} \right\|_\infty.
\end{aligned}$$

That is,

$$\left\| \frac{\rho_K}{\rho_E} \right\|_\infty \geq \left\| \frac{\rho_K}{\rho_{\bar{L}_\infty K}} \right\|_\infty,$$

which implies that $\bar{L}_\infty K$ is a solution to the desired extremal problem.

Assume that E_0 is a solution to the considered extremal problem. Now, we aim to show that $\left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty E_0$ is an origin-symmetric ellipsoid of minimal volume containing K . If so, according to the uniqueness of $L_\infty K$, we obtain that $\bar{L}_\infty K$ is the unique solution to the considered problem.

Let $E' \in \mathcal{E}^n$ with $K \subseteq E'$. From the facts that $1 \geq \|\rho_K/\rho_{E'}\|_\infty$ and $V(E_0) = \omega_n$, it follows that

$$V(E') \geq \left\| \frac{\rho_K}{\rho_{E'}} \right\|_\infty^n V(E') = \frac{V(E_0)}{\omega_n} \left\| \frac{\rho_K}{\rho_{E'}} \right\|_\infty^n V(E') = \left\| \frac{\rho_K}{\rho_{\left(\frac{\omega_n}{V(E')}\right)^{\frac{1}{n}} E'}} \right\|_\infty^n V(E_0).$$

Especially, we have

$$V(E') \geq \left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty^n V(E_0) = V\left(\left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty E_0\right).$$

Note that $K \subseteq \left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty E_0$. Thus, the ellipsoid $\left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty E_0$ is an origin-symmetric ellipsoid of minimal volume containing K . \square

Now, we turn to the main result in this section.

Theorem 6.4. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then $\lim_{p \rightarrow \infty} L_{\varphi^p} K = L_\infty K$.*

From the arguments in Section 5, we know that the set $\{\overline{L}_{\varphi^p}K : 1 \leq p < \infty\}$ is bounded from above. Hence, there exists a constant $C \in (0, \infty)$ such that

$$\begin{aligned} & \{\overline{L}_{\varphi^p}K : 1 \leq p < \infty\} \cup \{\overline{L}_\infty K\} \\ & \subseteq \mathcal{F} = \{E \in \mathcal{E}^n : V(E) = \omega_n \text{ and } E \subseteq CB\}. \end{aligned}$$

For $p \in [1, \infty)$, define the functional $f_p : \mathcal{F} \rightarrow (0, \infty)$ by

$$f_p(E) = O_{\varphi^p}(K, E), \quad \text{for } E \in \mathcal{F},$$

and the functional $f_\infty : \mathcal{F} \rightarrow (0, \infty)$ by

$$f_\infty(E) = \left\| \frac{\rho_K}{\rho_E} \right\|_\infty, \quad \text{for } E \in \mathcal{F}.$$

To prove Theorem 6.4, several lemmas are in order.

First, applying Lemma 3.9 (4) to the functionals f_j and f_∞ on \mathcal{F} , we have

Lemma 6.5. $\lim_{j \rightarrow \infty} f_j(E) = f_\infty(E)$, for $E \in \mathcal{F}$.

Lemma 6.6. The limit $\lim_{j \rightarrow \infty} f_j(E) = f_\infty(E)$ is uniform in $E \in \mathcal{F}$.

Proof. We argue by contradiction and assume the conclusion to be false. By our assumption, the definitions of f_{j_k} and f_∞ together with Lemma 3.9 (3), there exist an $\varepsilon_0 > 0$, a sequence $\{j_k\}_k$ strictly increasing to ∞ , and a sequence $E_k \in \mathcal{F}$, such that

$$|f_\infty(E_k) - f_{j_k}(E_k)| > \varepsilon_0; \quad \text{i.e.,} \quad f_{j_k}(E_k) < f_\infty(E_k) - \varepsilon_0, \quad \text{for } k \in \mathbb{N}.$$

Thus, these inequalities together with Lemma 3.9 (3) yield that

$$f_i(E_k) < f_\infty(E_k) - \varepsilon_0, \quad \text{for } i \leq j_k \text{ and } k \in \mathbb{N}.$$

Meanwhile, from the compactness of (\mathcal{F}, δ_H) together with Lemma 2.3, there exists a convergent subsequence $\{E_{k_l}\}_l$ of $\{E_k\}_k$, which converges to certain $E_0 \in \mathcal{F}$.

Consequently, letting $l \rightarrow \infty$ in the inequality

$$f_i(E_{k_l}) < f_\infty(E_{k_l}) - \varepsilon_0, \quad \text{for } i \leq k_l \text{ and } l \in \mathbb{N},$$

and using the continuity of f_i and f_∞ , we have

$$f_i(E_0) \leq f_\infty(E_0) - \varepsilon_0, \quad \text{for } i \in \mathbb{N},$$

which contradicts Lemma 6.5. □

Using Lemma 6.6, we can prove the following.

Lemma 6.7. $\lim_{p \rightarrow \infty} \overline{L}_{\varphi^p}K = \overline{L}_\infty K$.

Proof. By the boundedness of $\{\bar{L}_{\varphi^p} K : 1 \leq p < \infty\}$ in (\mathcal{F}, δ_H) , it suffices to prove that

$$\lim_{j \rightarrow \infty} \bar{L}_{\varphi^{p_j}} K = \bar{L}_\infty K,$$

for any convergent subsequence $\{\bar{L}_{\varphi^{p_j}} K\}_j$ with p_j strictly increasing to ∞ .

Assume that $\lim_{j \rightarrow \infty} \bar{L}_{\varphi^{p_j}} K = E_0$. From the definition of f_∞ , the continuity of f_∞ , Lemma 6.5, and Lemma 6.6, we have

$$\begin{aligned} \left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty &= f_\infty(E_0) \\ &= f_\infty \left(\lim_{j \rightarrow \infty} \bar{L}_{\varphi^{p_j}} K \right) \\ &= \lim_{j \rightarrow \infty} f_\infty(\bar{L}_{\varphi^{p_j}} K) \\ &= \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f_{p_i}(\bar{L}_{\varphi^{p_j}} K) \\ &= \lim_{i, j \rightarrow \infty} f_{p_i}(\bar{L}_{\varphi^{p_j}} K) \\ &= \lim_{j \rightarrow \infty} f_{p_j}(\bar{L}_{\varphi^{p_j}} K). \end{aligned}$$

Moreover, from the definition of $\bar{L}_{\varphi^{p_j}} K$ together with the fact that $\bar{L}_{\varphi^{p_j}} K \in \mathcal{F}$, Lemma 6.6 together with the compactness of \mathcal{F} , Lemma 6.5, and the definition of f_∞ , it follows

$$\begin{aligned} \left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty &= \lim_{j \rightarrow \infty} \min_{E \in \mathcal{F}} f_{p_j}(E) \\ &= \min_{E \in \mathcal{F}} \lim_{j \rightarrow \infty} f_{p_j}(E) \\ &= \min_{E \in \mathcal{F}} f_\infty(E) \\ &= \min_{E \in \mathcal{F}} \left\| \frac{\rho_K}{\rho_E} \right\|_\infty. \end{aligned}$$

Thus,

$$\left\| \frac{\rho_K}{\rho_{E_0}} \right\|_\infty = \min_{E \in \mathcal{F}} \left\| \frac{\rho_K}{\rho_E} \right\|_\infty,$$

From the fact that $\bar{L}_\infty K \in \mathcal{F}$ and the uniqueness of $\bar{L}_\infty K$, it yields that $E_0 = \bar{L}_\infty K$. \square

Lemma 6.8. $\lim_{p \rightarrow \infty} O_{\varphi^p}(K, \bar{L}_{\varphi^p} K) = \left\| \frac{\rho_K}{\rho_{\bar{L}_\infty K}} \right\|_\infty.$

Proof. From the definition of f_p , Lemma 6.6, Lemma 6.7, and the definition of f_∞ , it follows that

$$\begin{aligned} \lim_{p \rightarrow \infty} O_{\varphi^p}(K, \bar{L}_{\varphi^p} K) &= \lim_{p \rightarrow \infty} f_p(\bar{L}_{\varphi^p} K) \\ &= \left(\lim_{p \rightarrow \infty} f_p \right) \left(\lim_{p \rightarrow \infty} \bar{L}_{\varphi^p} K \right) \\ &= f_\infty(\bar{L}_\infty K) \\ &= \left\| \frac{\rho_K}{\rho_{\bar{L}_\infty K}} \right\|_\infty, \end{aligned}$$

as desired. \square

Now, we are in the position to finish the proof of Theorem 6.4.

Proof of Theorem 6.4. From the identities

$$O_{\varphi^p}(K, \bar{L}_{\varphi^p} K) \bar{L}_{\varphi^p} K = L_{\varphi^p} K \quad \text{and} \quad \left\| \frac{\rho_K}{\rho_{\bar{L}_\infty K}} \right\|_\infty \bar{L}_\infty K = L_\infty K,$$

together with Lemmas 6.7 and 6.8, Theorem 6.4 is derived immediately. \square

Note that if K is an origin-symmetric star body in \mathbb{R}^n , then the Orlicz-Legendre ellipsoid $L_{\varphi^p} K$ converges to the L wner ellipsoid LK as $p \rightarrow \infty$.

7. A Characterization of Orlicz-Legendre ellipsoid

In this section, we establish a connection linking the characterization of Orlicz-Legendre ellipsoids and the isotropy of measures.

Definition 7.1. Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi \cap C^1[0, \infty)$, the Borel measure $\mu_\varphi(K, \cdot)$ on S^{n-1} is defined by

$$d\mu_\varphi(K, \cdot) = \varphi'(\rho_K) \rho_K^{n+1} dS.$$

The next theorem not only characterizes the ellipsoid $L_\varphi^\diamond K$, but also plays a crucial role to establish Theorem 7.4.

Theorem 7.2. Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi \cap C^1[0, \infty)$. Then, $L_\varphi^\diamond K = B$, if and only if the measure $\mu_\varphi(K, \cdot)$ is isotropic on S^{n-1} , i.e.,

$$\frac{n}{|\mu_\varphi(K, \cdot)|} \int_{S^{n-1}} u \otimes u d\mu_\varphi(K, u) = I_n.$$

Proof. First, We show the necessity by variational method.

Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Choose $\varepsilon_0 > 0$ sufficiently small so that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the matrix $I_n + \varepsilon L$ is invertible. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, define

$$L_\varepsilon = \frac{I_n + \varepsilon L}{|I_n + \varepsilon L|^{\frac{1}{n}}}.$$

Then $L_\varepsilon \in \text{SL}(n)$. The assumption that $L_\varphi^\diamond K = B$ implies that for all ε ,

$$\tilde{V}_\varphi(K, L_\varepsilon^{-1}B) \geq \tilde{V}_\varphi(K, B).$$

The fact $\frac{1}{\rho_{L_\varepsilon^{-1}B}(u)} = h_{L_\varepsilon^t B}(u)$ for $u \in S^{n-1}$, together with the definition of $\tilde{V}_\varphi(K, L_\varepsilon^{-1}B)$, gives

$$\tilde{V}_\varphi(K, L_\varepsilon^{-1}B) = \int_{S^{n-1}} \varphi \left(\rho_K(u) \frac{(1 + 2\varepsilon u \cdot Lu + \varepsilon^2 Lu \cdot Lu)^{\frac{1}{2}}}{|I_n + \varepsilon L|^{\frac{1}{n}}} \right) d\tilde{V}_K(u).$$

From the smoothness of φ and $|L_\varepsilon u|$ in ε , the integrand depends smoothly on ε . Thus,

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \tilde{V}_\varphi(K, L_\varepsilon^{-1}B) = 0.$$

Calculating it directly, we have

$$\begin{aligned} 0 &= \int_{S^{n-1}} \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \varphi \left(\rho_K(u) \frac{(1 + 2\varepsilon u \cdot Lu + \varepsilon^2 Lu \cdot Lu)^{\frac{1}{2}}}{|I_n + \varepsilon L|^{\frac{1}{n}}} \right) d\tilde{V}_K(u) \\ &= \int_{S^{n-1}} \varphi'(\rho_K(u)) \left(-\frac{\text{tr } L}{n} + u \cdot Lu \right) \rho_K(u) d\tilde{V}_K(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \left(-\frac{\text{tr } L}{n} + u \cdot Lu \right) d\mu_\varphi(K, u). \end{aligned}$$

Let $v \in S^{n-1}$ and $L = v \otimes v$. Using the facts $\text{tr}(v \otimes v) = 1$ and $u \cdot (v \otimes v)u = (u \cdot v)^2$, it gives

$$\int_{S^{n-1}} (u \cdot v)^2 d\mu_\varphi(K, u) = \frac{|\mu_\varphi(K, \cdot)|}{n}.$$

Thus, $\mu_\varphi(K, \cdot)$ is isotropic on S^{n-1} .

Next, we prove the sufficiency. Suppose that $\mu_\varphi(K, \cdot)$ is isotropic on S^{n-1} . It suffices to prove that if $E \in \mathcal{E}^n$ and $V(E) = \omega_n$, then

$$\tilde{V}_\varphi(K, E) \geq \tilde{V}_\varphi(K, B),$$

If so, it will imply that $L_\varphi^\diamond K = B$. The proof will be completed after three steps.

First, for $a = (a_1, \dots, a_n) \in [0, \infty)^n$, define

$$F(a) = \int_{S^{n-1}} \varphi(\rho_K(u)) |\text{diag}(a_1, \dots, a_n)u| d\tilde{V}_K(u),$$

where $\text{diag}(a_1, \dots, a_n)$ denotes the $n \times n$ diagonal matrix with diagonal elements a_1, \dots, a_n .

We aim to show that

$$(7.1) \quad F(a) \geq F(e), \quad \text{whenever } \prod_{j=1}^n a_j = 1.$$

Here, e denotes the point $(1, \dots, 1)$.

From the smoothness of φ and $|\text{diag}(a_1, \dots, a_n)u|$ in (a_1, \dots, a_n) , we have

$$\begin{aligned} \frac{\partial}{\partial a_j} \Big|_{a=e} F(a) &= \int_{S^{n-1}} \frac{\partial}{\partial a_j} \Big|_{a=e} \varphi(\rho_K(u)) |\text{diag}(a_1, \dots, a_n)u| d\tilde{V}_K(u) \\ &= \int_{S^{n-1}} \varphi'(\rho_K(u)) \rho_K(u) \frac{\partial}{\partial a_j} \Big|_{a=e} |\text{diag}(a_1, \dots, a_n)u| d\tilde{V}_K(u) \\ &= \int_{S^{n-1}} u_j^2 \varphi'(\rho_K(u)) \rho_K(u) d\tilde{V}_K(u), \end{aligned}$$

where (u_1, \dots, u_n) denotes the coordinates of $u \in S^{n-1}$. From the isotropy of $\mu_\varphi(K, \cdot)$, it follows that

$$\frac{\partial}{\partial a_j} \Big|_{a=e} F(a) = \frac{|\mu_\varphi(K, \cdot)|}{n}.$$

Thus,

$$(7.2) \quad \nabla F(e) = \frac{|\mu_\varphi(K, \cdot)|}{n} e.$$

It can be checked that the function $F : [0, \infty)^n \rightarrow [0, \infty)$ is continuous and convex, and $F(\lambda a)$ is strictly increasing in $\lambda \in [0, \infty)$, for $a \in (0, \infty)^n$. Thus, $F^{-1}([0, F(e)])$ is compact, convex and of non-empty interior. Precisely, it is a convex body. Its boundary is given by the equation $F(a) = F(e)$ with $a \in [0, \infty)^n$, so (7.2) implies the vector e is an outer normal of the convex body $F^{-1}([0, F(e)])$ at the boundary point e .

Consequently, $F^{-1}([0, F(e)]) \subset \{a \in \mathbb{R}^n : a \cdot e \leq n\}$. That is to say, for all $a \in [0, \infty)^n$, if $F(a) \leq F(e)$, then $a \cdot e \leq n$. In contrast, for all $b = (b_1, \dots, b_n) \in (0, \infty)^n$ with $b_1 \cdots b_n = 1$, the AM-GM inequality yields that $b \cdot e \geq n$, with equality if and only if $b = e$. Hence, (7.1) is derived.

Secondly, with (7.1) in hand, we aim to show that for $T \in \text{SL}(n)$,

$$(7.3) \quad \tilde{V}_\varphi(TK, B) \geq \tilde{V}_\varphi(K, B),$$

with equality if and only if T is orthogonal.

Indeed, it is known that each $T \in \text{SL}(n)$ can be represented as $T^{-1} = O_1^{-1} A O_2$, where O_1, O_2 are $n \times n$ orthogonal matrices, and $A = \text{diag}(a_1, \dots, a_n)$ is diagonal and positive definite with $a_1 a_2 \cdots a_n = 1$. Note that $\tilde{V}_\varphi(TK, B) = \tilde{V}_\varphi(O_1 K, AB)$. So, applying (7.1) to the body $O_1 K$, it gives (7.3).

Finally, we rewrite inequality (7.3) equivalently as

$$\tilde{V}_\varphi(K, E) \geq \tilde{V}_\varphi(K, B),$$

for all $E \in \mathcal{E}^n$ with $V(E) = \omega_n$, with equality if and only if $E = B$. So, $L_\varphi^\circ K = B$.

The proof is complete. \square

Corollary 7.3. *Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi \cap C^1[0, \infty)$. Then, modulo orthogonal transformations, there exists an $SL(n)$ transformation T such that the measure $\mu_\varphi(TK, \cdot)$ is isotropic on S^{n-1} .*

Theorem 7.4. *Suppose $K \in \mathcal{S}_o^n$, $\varphi \in \Phi \cap C^1[0, \infty)$ and $T \in SL(n)$. Then the following assertions are equivalent:*

- (1) $L_\varphi(O_\varphi(K, TB)^{-1}K) = TB$,
- (2) $\bar{L}_\varphi K = TB$,
- (3) $L_\varphi^\circ(O_\varphi(T^{-1}K, B)^{-1}T^{-1}K) = B$,
- (4) $\mu_\varphi(O_\varphi(T^{-1}K, B)^{-1}T^{-1}K, \cdot)$ is isotropic on S^{n-1} .

Proof. Equations (4.3) yields the equivalence “(1) \Leftrightarrow (2)”. Combining Lemma 4.5 with Lemma 3.4 (1), it gives the equivalence “(2) \Leftrightarrow (3)”. Finally, Theorem 7.2 implies the equivalence “(3) \Leftrightarrow (4)”. \square

8. VOLUME RATIO INEQUALITIES

In general, the Orlicz-Legendre ellipsoid $L_\varphi K$ does not contain K . However, we show that the volume functional over the class of Orlicz-Legendre ellipsoids of K is bounded by $V(L_1 K)$ from below and by $V(L_\infty K)$ from above.

Theorem 8.1. *Suppose $K \in \mathcal{S}_o^n$, $\varphi \in \Phi$ and $1 \leq p < q < \infty$. Then*

$$V(L_1 K) \leq V(L_\varphi K) \leq V(L_{\varphi^p} K) \leq V(L_{\varphi^q} K) \leq V(L_\infty K).$$

Proof. From Lemma 3.9, it follows that

$$\begin{aligned} \left\{ E \in \mathcal{E}^n : \left\| \frac{\rho_K}{\rho_E} : V_K^* \right\|_1 \leq 1 \right\} &\supseteq \left\{ E \in \mathcal{E}^n : \left\| \frac{\rho_K}{\rho_E} : V_K^* \right\|_\varphi \leq 1 \right\} \\ &\supseteq \left\{ E \in \mathcal{E}^n : \left\| \frac{\rho_K}{\rho_E} : V_K^* \right\|_{\varphi^p} \leq 1 \right\} \\ &\supseteq \left\{ E \in \mathcal{E}^n : \left\| \frac{\rho_K}{\rho_E} : V_K^* \right\|_{\varphi^q} \leq 1 \right\} \\ &\supseteq \left\{ E \in \mathcal{E}^n : \left\| \frac{\rho_K}{\rho_E} \right\|_\infty \leq 1 \right\}. \end{aligned}$$

From the above inclusions and the definition of Orlicz-Legendre ellipsoids, the desired inequalities are obtained. \square

Theorem 8.2. Suppose $K \in \mathcal{S}_o^n$ and $\varphi \in \Phi$. Then

$$V(L_\varphi K) \geq V(K),$$

with equality if and only if $K \in \mathcal{E}^n$.

Proof. From Lemma 3.6, it follows that

$$O_\varphi(K, L_\varphi K) \geq \left(\frac{V(K)}{V(L_\varphi K)} \right)^{\frac{1}{n}},$$

with equality if and only if $K \in \mathcal{E}^n$. Combing this with the fact

$$1 = O_\varphi(K, L_\varphi K),$$

the desired inequality is followed. \square

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then Theorem 8.2 implies that $V(L_p K) \geq V(K)$, and in particular that $V(\Gamma_2 K) \geq V(K)$.

A classical result on John's ellipsoid is Ball's volume ratio inequality [1, 2], which states: if K is an origin-symmetric convex body in \mathbb{R}^n , then

$$\frac{V(K)}{V(JK)} \leq \frac{2^n}{\omega_n},$$

with equality if and only if K is a parallelotope. The fact that equality holds in Ball's inequality only for parallelotope was established by Barthe [3]. He also established the outer volume-ratio inequality: if K is an origin-symmetric convex body in \mathbb{R}^n , then

$$\frac{V(K)}{V(LK)} \geq \frac{2^n}{n! \omega_n},$$

with equality if and only if K is a cross-polytope.

Recall that when K is an origin-symmetric convex body, $L_\infty K$ is just the Löwner ellipsoid LK . Thus, Combining Theorem 8.1 with Barthe's outer volume ratio inequality, we immediately obtain

Theorem 8.3. Suppose $K \in \mathcal{K}_o^n$ is origin-symmetric and $\varphi \in \Phi$. Then

$$\frac{V(K)}{V(L_\varphi K)} \geq \frac{2^n}{n! \omega_n}.$$

It is easily seen that the volume ratio $\frac{V(L_\varphi K)}{V(K)}$ is $\text{GL}(n)$ -invariant and minimized by origin-symmetric ellipsoids. Theorem 8.3 shows that $\frac{V(L_\varphi K)}{V(K)}$ is bounded from above. However, the exact equality condition is not yet known.

Problem. Suppose $\varphi \in \Phi$. Amongst all origin-symmetric convex bodies K in \mathbb{R}^n , which ones maximize the volume ratio $\frac{V(L_\varphi K)}{V(K)}$?

A particular case concerns with the volume ratio $\frac{V(\Gamma_2 K)}{V(K)}$. As pointed out by Schneider [59] and LYZ [43], that to find the maximizers for $\frac{V(\Gamma_2 K)}{V(K)}$ over the class of origin-symmetric convex bodies is still a major open problem in convex geometry. It is even difficult to show that there exists a constant c which is independent of the dimension n and bounds the volume ratio $\frac{V(\Gamma_2 K)}{V(K)}$ from above. This problem was firstly posed by Bourgain [9]. For more information, we refer to Bourgain [10], Dar [13], Junge [31], Lindenstrauss and Milman [34], LYZ [43], and Milman and Pajor [53].

APPENDIX A

Lemma A.1. *Suppose $\{T_j\}_{j \in \mathbb{N}} \subset \text{SL}(n)$. Then*

$$\|T_j\| \rightarrow \infty \iff \|T_j^{-1}\| \rightarrow \infty.$$

Thus, $\{T_j\}_{j \in \mathbb{N}}$ is bounded from above, if and only if $\{T_j^{-1}\}_{j \in \mathbb{N}}$ is bounded from above.

Proof. It suffices to prove the implication

$$(8.1) \quad \|T_j\| \rightarrow \infty \implies \|T_j^{-1}\| \rightarrow \infty.$$

For this aim, represent any $T \in \text{SL}(n)$ in the form $T = O_1 A O_2$, where O_1, O_2 are $n \times n$ orthogonal matrices, and $A = \text{diag}(a_1, \dots, a_n)$ is an $n \times n$ diagonal matrix, with positive diagonal elements a_1, \dots, a_n , and $\det(A) = 1$. Then,

$$(8.2) \quad \|T\| = \max_{1 \leq i \leq n} a_i.$$

Observe that $T^{-1} = O_2^{-1} A^{-1} O_1^{-1}$, the matrices O_2^{-1}, O_1^{-1} are orthogonal, and $A^{-1} = \text{diag}\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right)$. Thus,

$$(8.3) \quad \|T^{-1}\| = \max_{1 \leq i \leq n} \frac{1}{a_i} = \frac{1}{\min_{1 \leq i \leq n} a_i}.$$

Meanwhile, the condition $\prod_{i=1}^n a_i = 1$ together with the inequality

$$\left(\min_{1 \leq i \leq n} a_i\right)^{n-1} \max_{1 \leq i \leq n} a_i \leq \prod_{i=1}^n a_i$$

gives

$$(8.4) \quad \frac{1}{\min_{1 \leq i \leq n} a_i} \geq \left(\max_{1 \leq i \leq n} a_i\right)^{\frac{1}{n-1}}.$$

Hence, by (8.2), (8.3) and (8.4), the implication (8.1) is derived. □

Lemma A.2. Suppose $\{T_j\}_{j \in \mathbb{N}} \subset \text{SL}(n)$, $T_0 \in \text{SL}(n)$. If $T_j \rightarrow T_0$ with respect to d_n , then

- (1) $T_j^t B \rightarrow T_0^t B$ with respect to δ_H .
- (2) $T_j^{-1} \rightarrow T_0^{-1}$ with respect to d_n .
- (3) $T_j B \rightarrow T_0 B$ with respect to $\tilde{\delta}_H$.

Proof. From the following implications,

$$\begin{aligned}
\|T_j - T_0\| \rightarrow 0 &\iff T_j u \rightarrow T_0 u, \text{ uniformly for } u \in S^{n-1}, \\
&\implies |T_j u| \rightarrow |T_0 u| \text{ uniformly, for } u \in S^{n-1}, \\
&\iff h_{T_j^t B}(u) \rightarrow h_{T_0^t B}(u), \text{ uniformly for } u \in S^{n-1}, \\
&\iff T_j^t B \rightarrow T_0^t B, \text{ with respect to } \delta_H,
\end{aligned}$$

it yields (1) directly.

Since $T_j \rightarrow T_0$, the sequence $\{T_j\}$ is bounded in (\mathcal{L}^n, d_n) . By Lemma A.1, the sequence $\{T_j^{-1}\}$ is also bounded. Thus, to prove $T_j^{-1} \rightarrow T_0^{-1}$, it suffices to prove any convergent subsequence $\{T_{j_k}^{-1}\}_{k \in \mathbb{N}}$ converges to T_0^{-1} . Assume $T_{j_k}^{-1} \rightarrow T$.

Since $\|T_{j_k} - T_0\| \rightarrow 0$ and $\|T_{j_k}^{-1} - T\| \rightarrow 0$, from $\sup_{k \in \mathbb{N}} \|T_{j_k}\| < \infty$, we have

$$\begin{aligned}
\|T_{j_k} T_{j_k}^{-1} - T_0 T\| &\leq \|T_{j_k} (T_{j_k}^{-1} - T)\| + \|(T_{j_k} - T_0) T\| \\
&\leq \|T_{j_k}^{-1} - T\| \|T_{j_k}\| + \|T_{j_k} - T_0\| \|T\| \\
&\leq \|T_{j_k}^{-1} - T\| \sup_{k \in \mathbb{N}} \|T_{j_k}\| + \|T_{j_k} - T_0\| \|T\|,
\end{aligned}$$

so, it concludes that $T_{j_k} T_{j_k}^{-1} \rightarrow T_0 T$. Since $T_{j_k} T_{j_k}^{-1} = I_n, \forall k$, it follows that $T = T_0^{-1}$.

That $T_j^{-1} \rightarrow T_0^{-1}$ with respect to d_n implies that

$$|T_j^{-1} u| \rightarrow |T_0^{-1} u|; \quad \text{i.e.,} \quad \rho_{T_j B}(u) \rightarrow \rho_{T_0 B}(u), \quad \text{uniformly for } u \in S^{n-1}.$$

Thus, $T_j B \rightarrow T_0 B$ with respect to $\tilde{\delta}_H$. □

Lemma A.3. Suppose $E_0 \in \mathcal{E}^n$, $\{E_j\}_{j \in \mathbb{N}} \subset \mathcal{E}^n$ and $V(E_j) = a, \forall j \in \mathbb{N}, a > 0$. Then $E_j \rightarrow E_0$ with respect to δ_H , if and only if $E_j \rightarrow E_0$ with respect to $\tilde{\delta}_H$.

Proof. Under the standard orthonormal basis, there exist unique symmetric and positive definite matrices T_0, T_j , such that $T_0 B = E_0$ and $T_j B = E_j$.

We first prove the following implication:

$$(8.5) \quad T_j B \rightarrow T_0 B, \text{ with respect to } \delta_H. \implies T_j \rightarrow T_0, \text{ with respect to } d_n.$$

That $T_j B \rightarrow T_0 B$ with respect to δ_H implies that $\sup\{\|T_j\| : j \in \mathbb{N}\} < \infty$. Thus, to prove $T_j \rightarrow T_0$, it suffices to prove any convergent subsequence $\{T_{j_k}\}_{k \in \mathbb{N}}$ of $\{T_j\}$ converges to T_0 . Assume $T_{j_k} \rightarrow T$. Let $T_j = (t_{l,m}^j)_{1 \leq l, m \leq n}$ and $T = (t_{l,m})_{1 \leq l, m \leq n}$. By our assumption, $x \cdot T_{j_k} y \rightarrow x \cdot T y$, for $x \in \mathbb{R}^n$.

Now, three observations are in order. First, $t_{l,m}^j \rightarrow t_{l,m}$, $\forall(l, m)$. Thus, the symmetry of each T_j implies the symmetry of T ; Second, $\det(T) = 1$. Indeed, since $\det(T_j)$ is a continuous function of the elements of T_j , from the fact $t_{l,m}^j \rightarrow t_{l,m}$ for each (l, m) , and the fact $\det(T_j) = 1$ for each j , we obtain $\det(T) = 1$; Third, T is positive definite. Indeed, from $x \cdot T_j x \rightarrow x \cdot T x$, together with the positive definiteness of each T_j , we know that T is positive semi-definite. Added that $\det(T) = 1$, it follows that T is positive definite.

Since $T_{j_k} \rightarrow T$, by Lemma A.2 (1), it follows that $T_{j_k} B \rightarrow T B$ with respect to δ_H . Thus, $T B = T_0 B$. Since T_0 is the unique symmetric positive definite matrix such that $T_0 B = E_0$, it concludes that $T = T_0$. Thus, $T_j \rightarrow T_0$.

Now, from implication (8.5) and Lemma A.2 (1), it yields the implication

$$E_j \rightarrow E_0 \text{ with respect to } \delta_H. \implies E_j \rightarrow E_0 \text{ with respect to } \tilde{\delta}_H.$$

Conversely, assume that $E_j \rightarrow E_0$ with respect to $\tilde{\delta}_H$; i.e., $\rho_{E_j} \rightarrow \rho_{E_0}$, uniformly on S^{n-1} . From (2.3) and the equation

$$|h_{E_j^*}(u) - h_{E_0^*}(u)| = \frac{|\rho_{E_j}(u) - \rho_{E_0}(u)|}{\rho_{E_j}(u)\rho_{E_0}(u)}, \quad \text{for } u \in S^{n-1},$$

it follows that $h_{E_j^*} \rightarrow h_{E_0^*}$, uniformly on S^{n-1} , i.e., $E_j^* \rightarrow E_0^*$ with respect to δ_H . Note that $E_j^* = T_j^{-1} B$, $E_0^* = T_0^{-1} B$, and T_j^{-1} , T_0^{-1} are both symmetric positive definite. From implication (8.5), it concludes that $T_j^{-1} \rightarrow T_0^{-1}$. Thus, By Lemma A.2 (2), $T_j \rightarrow T_0$. Therefore, by Lemma A.2 (1), $T_j B \rightarrow T_0 B$ with respect to δ_H . \square

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